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Geodesic Mappings of Conformally Flat and Three-Dimensional Pseudo-Riemannian Spaces

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Abstract

Geodesic mappings constitute an important topic in differential geometry and play a significant role in mathematical physics, especially in the geometric study of gravitational and other field theories. In this paper, we investigate geodesic mappings of conformally flat (*pseudo*)-Riemannian spaces of dimension $n > 3$ and of arbitrary three-dimensional (*pseudo*)-Riemannian spaces.

Using a lemma due to Moldobayev concerning the structure of solutions of the Sinyukov equations, we show that the fundamental tensor associated with a geodesic mapping can be represented as a linear combination of the metric tensor and the Ricci tensor. By means of the Einstein tensor, we derive a new intrinsic characterization of conformally flat spaces and three-dimensional spaces admitting nontrivial geodesic mappings.

In particular, we obtain a recurrent-type differential condition for the Einstein tensor and show that this condition is necessary and sufficient under natural gradient assumptions. The obtained results provide internal criteria for the existence of nontrivial geodesic mappings and clarify the geometric structure of such spaces.

Keywords: pseudo-Riemannian space, geodesic mapping, conformally flat space, Einstein tensor, Sinyukov equations.
MSC 2020: 53B20, 53B21.

1. Introduction

In a number of papers and books [1, 2, 4, 5, 6, 7, 9, 13], geodesic mappings of many classes of n-dimensional (*pseudo*)-Riemannian spaces $\mathbb{V}_n$ have been investigated, including conformally flat spaces. A conformally flat space is a space for which the Weyl conformal curvature tensor vanishes. It is well known [2, 6, 13] that conformally Euclidean spaces $\mathbb{V}_n$ and all three-dimensional spaces $\mathbb{V}_3$ are conformally flat.

Various problems concerning geodesic mappings have been studied.

In the works of Rosenfeld [10, 11], necessary conditions were obtained for conformally flat spaces $\mathbb{V}_n$ $(n > 2)$ admitting nontrivial geodesic mappings onto arbitrary (*pseudo*)-Riemannian spaces $\bar{\mathbb{V}}_n$. It was also shown that subprojective spaces of the main type are the only conformally flat spaces $\mathbb{V}_n$ $(n > 3)$ admitting nontrivial geodesic mappings onto conformally flat spaces $\bar{\mathbb{V}}_n$.

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In Moldobayev's dissertation [8], a remarkable formula was proved showing that, in these spaces $\mathbb{V}_n$, the fundamental solutions of the Sinyukov equations, see [5, 13], are linear combinations of the metric tensor and the Ricci tensor. This result is also mentioned in Mikeš's review paper [4] and in the book [5]. It leads to a direct conclusion that the degree of mobility r_{gm} of three-dimensional spaces $\mathbb{V}_3$ and conformally flat spaces $\mathbb{V}_n$ of nonconstant curvature is $r_{gm} = 2$. For completeness, we present a detailed proof of this result.

Using these relations, we derive a new intrinsic characterization of the considered spaces.

2. Fundamental equations of geodesic mappings

Let $V_n = (M, g)$, $\bar{V}_n = (\bar{M}, \bar{g})$, be (*pseudo*)-Riemannian manifolds, where M and $\bar{M}$ are n-dimensional manifolds and g and $\bar{g}$ are metrics on $\mathbb{V}_n$ and $\bar{\mathbb{V}}_n$, respectively.

Definition 2.1. A diffeomorphism $f : \mathbb{V}_n \to \bar{\mathbb{V}}_n$ is called a *geodesic mapping* of $\mathbb{V}_n$ onto $\bar{\mathbb{V}}_n$ if f maps every geodesic in $\mathbb{V}_n$ onto a geodesic in $\bar{\mathbb{V}}_n$.

Let there exist a geodesic mapping $f : \mathbb{V}_n \to \bar{\mathbb{V}}_n$. Since f is a diffeomorphism, we may choose local coordinate charts on M and $\bar{M}$ such that, locally, f maps points to points with the same coordinates. Thus, locally, we may identify $\bar{M}$ with M.

A manifold $\mathbb{V}_n$ admits a geodesic mapping onto $\bar{V}_n$ if and only if the *Levi-Civita equations*

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{2.1}$$

hold for all tangent vector fields X, Y, where ψ is a differential one-form and ∇ and $\bar{\nabla}$ are the Levi-Civita connections of $\mathbb{V}_n$ and $\bar{\mathbb{V}}_n$, respectively.

If $\psi \equiv 0$, then f is an affine mapping, or equivalently, a trivial geodesic mapping.

In local coordinates, equations (2.1) have the form

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i,$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of $\mathbb{V}_n$ and $\bar{V}_n$, respectively, ψ_i are the components of ψ, and δ^h_i is the Kronecker delta.

A new approach to the proof of the Levi-Civita equations is presented in [3].

The Levi-Civita equations (2.1) are equivalent to

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}, \tag{2.2}$$

where "," denotes the covariant derivative with respect to the connection of $\mathbb{V}_n$.

It is known that

$$\psi_i = \partial_i \Psi, \qquad \Psi = \frac{1}{2(n+1)} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \qquad \partial_i = \frac{\partial}{\partial x^i}.$$

Sinyukov [12], see also [4, 5, 13], proved that the Levi-Civita equations (2.1) are equivalent to

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{2.3}$$

where

$$a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \qquad \lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha.$$

From (2.3) it follows that

$$\lambda_i = \partial_i \lambda, \qquad \lambda = \frac{1}{2} a_{\alpha\beta} g^{\alpha\beta}.$$

Conversely,

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi}\tilde{g}_{ij}, \qquad \Psi = \frac{1}{2}\ln\left|\frac{\det\tilde{g}}{\det g}\right|, \qquad \|\tilde{g}_{ij}\| = \|g^{i\alpha}g^{j\beta}a_{\alpha\beta}\|^{-1}. \tag{2.4}$$

The above formulas provide a criterion for geodesic mappings $V_n \to \bar{V}_n$, both locally and globally. They hold for $\mathbb{V}_n, \bar{\mathbb{V}}_n \in C^1$, that is, for metrics $g, \bar{g} \in C^1$.

If $\mathbb{V}_n$ and $\bar{\mathbb{V}}_n$ are of class C^2, then under a geodesic mapping $V_n \to \bar{V}_n$ the Riemann and Ricci tensors transform as follows:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik}, \qquad \bar{R}_{ij} = R_{ij} - (n-1)\psi_{ij},$$

where

$$\psi_{ij} = \psi_{i,j} - \psi_i\psi_j.$$

The Weyl tensor of projective curvature, defined by

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right),$$

is invariant under geodesic mappings.

3. Conformal mappings of (*pseudo*)-Riemannian spaces

A conformal mapping of $\mathbb{V}_n$ onto $\bar{\mathbb{V}}_n$ is geometrically characterized as a mapping preserving angles. Equivalently, it is described by the condition

$$\bar{g} = \mathrm{e}^{2\sigma}g,$$

which in local coordinates reads [2, 5, 6, 7, 9, 13]

$$\bar{g}_{ij}(x) = \mathrm{e}^{2\sigma(x)}g_{ij}(x), \tag{3.1}$$

where $\sigma(x)$ is a function and $g_{ij}(x)$ and $\bar{g}_{ij}(x)$ are the components of the metric tensors of $\mathbb{V}_n$ and $\bar{\mathbb{V}}_n$ in a common coordinate system.

In a space $\mathbb{V}_n$ of class C^2, the Weyl conformal curvature tensor is defined by

$$C_{hijk} = R_{hijk} - g_{hk}L_{ij} - g_{ij}L_{hk} + g_{hj}L_{ik} + g_{ik}L_{hj}, \qquad C^h_{ijk} = g^{h\alpha}C_{\alpha ijk}. \tag{3.2}$$

The Weyl conformal curvature tensor is invariant under conformal mappings $V_n \to \bar{V}_n$, that is,

$$\bar{C}^h_{ijk} = C^h_{ijk},$$

where $\bar{C}^h_{ijk}$ is the Weyl conformal curvature tensor on $\bar{\mathbb{V}}_n$.

A space $\mathbb{V}_n$ is called conformally flat if there exists a conformal mapping of $\mathbb{V}_n$ onto a flat space. Since flat spaces are characterized by the vanishing of the curvature tensor R^h_{ijk}, conformally flat spaces have vanishing Weyl conformal curvature tensor:

$$C^h_{ijk} = 0. \tag{3.3}$$

Condition (3.3) is equivalent to the Riemann curvature tensor satisfying

$$R_{hijk} = g_{hk}L_{ij} + g_{ij}L_{hk} - g_{hj}L_{ik} - g_{ik}L_{hj}. \tag{3.4}$$

For $n > 3$, this condition is necessary and sufficient for conformal flatness. For $n = 3$, the Weyl conformal curvature tensor vanishes identically. In this case, conformal flatness is characterized by the additional condition

$$L_{ij,k} = L_{ik,j}.$$

In what follows, we investigate geodesic mappings of spaces $\mathbb{V}_n$ whose Weyl conformal curvature tensor vanishes. Thus, we consider conformally flat spaces $\mathbb{V}_n$ for $n > 3$ and arbitrary three-dimensional spaces $\mathbb{V}_3$.

4. Moldobayev's main lemma on geodesic mappings

The following lemma was formulated and proved in Moldobayev's dissertation [8] in the study of group properties of spaces. This result is also stated, with reference to Moldobayev's work, in Mikeš's review article [4]. For completeness, we present the original proof of this lemma.

Let $\mathbb{V}_n$ be a conformally flat space with $n > 3$, or let $\mathbb{V}_n$ be a three-dimensional (*pseudo*)-Riemannian space $\mathbb{V}_3$.

Lemma 4.1 (Moldobayev [8, p. 108])**.** Let $\mathbb{V}_n$ be a space of nonconstant curvature. Then every tensor a_{ij} satisfying equations (2.3) has the representation

$$a_{ij} = \alpha g_{ij} + \beta R_{ij}, \tag{4.1}$$

where α and β are functions and

$$\det(a_{ij}) \neq 0.$$

Proof. We consider equations (2.3) in a space $\mathbb{V}_n$ characterized by condition (3.4). The integrability conditions of (2.3) have the form

$$a_{i\alpha}R^{\alpha}_{jkl} + a_{\alpha j}R^{\alpha}_{ikl} = \lambda_{i,l}g_{jk} + \lambda_{j,l}g_{ik} - \lambda_{i,k}g_{jl} - \lambda_{j,k}g_{il}. \tag{4.2}$$

Using (3.4), we obtain

$$\begin{aligned} &a_{il}L_{jk} + a_{jl}L_{ik} - a_{ik}L_{jl} - a_{jk}L_{il} \\ &+a_{i\alpha}L^{\alpha}_{k}g_{jl} + a_{j\alpha}L^{\alpha}_{k}g_{il} - a_{i\alpha}L^{\alpha}_{l}g_{jk} - a_{j\alpha}L^{\alpha}_{l}g_{ik} \\ &= \lambda_{i,l}g_{jk} + \lambda_{j,l}g_{ik} - \lambda_{i,k}g_{jl} - \lambda_{j,k}g_{il}. \end{aligned} \tag{4.3}$$

In formula (4.3), we interchange the indices $i \leftrightarrow l$ and $j \leftrightarrow k$. We get

$$\begin{aligned} &a_{li}L_{kj} + a_{ki}L_{lj} - a_{lj}L_{ki} - a_{kj}L_{li} \\ &+a_{l\alpha}L^{\alpha}_{j}g_{ki} + a_{k\alpha}L^{\alpha}_{j}g_{li} - a_{l\alpha}L^{\alpha}_{i}g_{kj} - a_{k\alpha}L^{\alpha}_{i}g_{lj} \\ &= \lambda_{l,i}g_{kj} + \lambda_{k,i}g_{lj} - \lambda_{l,j}g_{ki} - \lambda_{k,j}g_{li}. \end{aligned} \tag{4.4}$$

Adding (4.3) and (4.4) term by term and using the symmetry of the tensors g_{ij}, a_{ij}, L_{ij}, and $a_{i\alpha}L^{\alpha}_{k}$, we obtain

$$a_{il}L_{jk} - a_{jk}L_{il} + a_{i\alpha}L^{\alpha}_{l}g_{jk} - a_{j\alpha}L^{\alpha}_{k}g_{il} = \lambda_{i,l}g_{jk} - \lambda_{j,k}g_{il}. \tag{4.5}$$

Here the symmetry of $a_{i\alpha}L^{\alpha}_{k}$ follows from

$$a_{i\alpha}R^{\alpha}_{k} = a_{k\alpha}R^{\alpha}_{i},$$

which is proved in [13].

Contracting (4.5) with g^{jk}, we find

$$\lambda_{i,l} - a_{i\alpha}L^{\alpha}_{l} = \frac{1}{n}\lambda^* g_{il} + \frac{1}{n}a^* L_{il} - \frac{1}{n}L^* a_{il}, \tag{4.6}$$

where λ^*, a^*, and L^* are functions.

Substituting (4.6) into (4.5), we obtain

$$\left(a_{il} - \frac{1}{n}a^* g_{il}\right)\left(L_{jk} - \frac{1}{n}L^* g_{jk}\right) = \left(a_{jk} - \frac{1}{n}a^* g_{jk}\right)\left(L_{il} - \frac{1}{n}L^* g_{il}\right). \tag{4.7}$$

Hence,

$$a_{il} - \frac{1}{n}a^* g_{il} \qquad \text{and} \qquad L_{il} - \frac{1}{n}L^* g_{il}$$

are proportional. Since

$$a_{il} - \frac{1}{n}a^* g_{il} \neq 0,$$

we obtain

$$a_{ij} = \alpha^{**} g_{ij} + \beta^{**} L_{ij},$$

where α^{**} and β^{**} are functions. Formula (4.1) follows from this representation. □

5. An intrinsic characterization of $\mathbb{V}_3$ and conformally flat spaces $\mathbb{V}_n$ admitting geodesic mappings

Let $\mathbb{V}_n$ be a conformally flat space with $n > 3$, or let $\mathbb{V}_n$ be a three-dimensional (*pseudo*)-Riemannian space $\mathbb{V}_3$. Assume that $\mathbb{V}_n$ admits a geodesic mapping onto $\bar{\mathbb{V}}_n$. Then, by Lemma 4.1, formula (4.1) holds. It is convenient to rewrite it in the form

$$a_{ij} = \alpha g_{ij} + \beta E_{ij}, \tag{5.1}$$

where the Einstein tensor is defined by

$$E_{ij} = R_{ij} - \frac{R}{n}g_{ij}, \tag{5.2}$$

and

$$R = R_{ij}g^{ij}$$

is the scalar curvature.

Differentiating formula (5.1) with respect to x^k and assuming that α, β, and E_{ij} are differentiable, we obtain

$$a_{ij,k} = \alpha_k g_{ij} + \beta_k E_{ij} + \beta E_{ij,k}.$$

From (2.3) we get

$$\lambda_i g_{jk} + \lambda_j g_{ik} = \alpha_k g_{ij} + \beta_k E_{ij} + \beta E_{ij,k}. \tag{5.3}$$

Evidently, $\beta \neq 0$. Moreover, β is chosen so that the determinant of the matrix (a_{ij}) does not vanish. Thus, at every point x, there is no neighbourhood in which the function $\beta(x)$ is identically zero.

Therefore, formula (5.3) can be written as

$$E_{ij,k} = \varphi_k E_{ij} + 2\theta_k g_{ij} + \frac{\lambda_i}{\beta} g_{jk} + \frac{\lambda_j}{\beta} g_{ik}, \tag{5.4}$$

where

$$\varphi_k = -\frac{\beta_k}{\beta}, \qquad 2\theta_k = -\frac{\alpha_k}{\beta}.$$

Contracting (5.4) with g^{ij}, we obtain

$$\frac{\lambda_k}{\beta} = -n\theta_k.$$

Thus formula (5.4) takes the form

$$E_{ij,k} = \varphi_k E_{ij} + 2\theta_k g_{ij} - n\theta_i g_{jk} - n\theta_j g_{ik}. \tag{5.5}$$

We now show that the vector fields φ_i and θ_i are determined by the intrinsic objects of the space $\mathbb{V}_n$, that is, by the metric tensor g and its derivatives. It is sufficient to prove that the homogeneous system

$$\varphi_k^* E_{ij} + 2\theta_k^* g_{ij} - n\theta_i^* g_{jk} - n\theta_j^* g_{ik} = 0 \tag{5.6}$$

has only the trivial solution with respect to the unknown vectors φ_i^* and θ_i^*.

Indeed, on $\mathbb{V}_n$ the Einstein tensor E_{ij} is nonzero; otherwise, the space would have constant curvature. Let vectors u_i and v^i satisfy

$$u_\alpha g^{\alpha\beta} E_{\beta i} = \rho u_i, \qquad v^\alpha u_\alpha = 0,$$

where $\rho \neq 0$ is an eigenvalue corresponding to the eigenvector u_i.

Contracting (5.6) with $u_\alpha g^{\alpha i}$, we obtain

$$(\rho\varphi_k^* + 2\theta_k^*)u_j - n u_\alpha g^{\alpha i}\theta_i^* g_{jk} - n\theta_j^* u_k = 0.$$

Since $\operatorname{rank}(g_{ij}) > 2$, it follows that

$$u_\alpha g^{\alpha i}\theta_i^* = 0.$$

Consequently,

$$(\rho\varphi_k^* + 2\theta_k^*)u_j - n\theta_j^* u_k = 0.$$

This implies

$$\varphi_i^* = c_1 u_i, \qquad \theta_i^* = c_2 u_i.$$

Substituting these expressions into (5.6), we get

$$c_1 u_k E_{ij} + 2c_2 u_k g_{ij} - n c_2 u_i g_{jk} - n c_2 u_j g_{ik} = 0.$$

After contracting with v^k, we obtain

$$c_2(u_i v_j + u_j v_i) = 0.$$

Hence $c_2 = 0$. Consequently,

$$c_1 u_k E_{ij} = 0,$$

and therefore $c_1 = 0$. Thus

$$\varphi_i^* = 0, \qquad \theta_i^* = 0.$$

We have proved the following theorem.

Theorem 5.1. *Let $\mathbb{V}_n$ be either a conformally flat (pseudo)-Riemannian space of dimension $n > 3$ or a three-dimensional (pseudo)-Riemannian space $\mathbb{V}_3$. Assume that $\mathbb{V}_n$ is of nonconstant curvature and admits a nontrivial geodesic mapping.*

Then the Einstein tensor satisfies the recurrent-type relation

$$E_{ij,k} = \varphi_k E_{ij} + 2\theta_k g_{ij} - n\theta_i g_{jk} - n\theta_j g_{ik}.$$

Conversely, this condition is sufficient provided that the following assumptions hold:

1. *The vector field φ_i is a gradient, that is,*

$$\varphi_i = \partial_i \Phi.$$

Then

$$\beta = c\mathrm{e}^{-\Phi},$$

where c is a constant.

2. *The vector field $\beta\theta_i$ is a gradient, that is,*

$$\beta\theta_i = \partial_i \Theta.$$

Then

$$\alpha = -2\Theta + \bar{c},$$

where $\bar{c}$ is a constant.

The constants c and $\bar{c}$ are chosen so that the tensor

$$a_{ij} = \alpha g_{ij} + \beta E_{ij}$$

is nondegenerate, that is,

$$\det(a_{ij}) \neq 0.$$

We note that metrics of three-dimensional spaces $\mathbb{V}_3$ of general signature admitting geodesic mappings were obtained by Petrov; see [1, 9].

6. Conclusion

In this paper, we studied geodesic mappings of conformally flat (*pseudo*)-Riemannian spaces of dimension $n > 3$ and of arbitrary three-dimensional (*pseudo*)-Riemannian spaces. Using Moldobayev's lemma, we showed that the fundamental tensor of the Sinyukov equations can be expressed as a linear combination of the metric tensor and the Ricci tensor.

This representation allowed us to formulate an intrinsic criterion for the existence of nontrivial geodesic mappings in terms of the Einstein tensor. In particular, we obtained a recurrent-type differential relation for the Einstein tensor and showed that, under natural gradient assumptions, this condition is also sufficient.

The result gives a new internal characterization of the considered spaces and clarifies the geometric role of the Einstein tensor in the theory of geodesic mappings.

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Definition and Construction of New Types of Kinematic Spaces with Assistance of Artificial Intelligence

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Abstract

The concept of "kinematic space" is interpreted as intended for computer implementation; the distance between two screen images is equal to the minimum transition time between them within the space. A survey of developing this idea and other results within the frames of "digital topology" since 1960-s is presented. Controlled transformations (controlled motion of points) are considered too. Basing on an example of controlled rotation in 4D-Euclidean space with the aim to make a mirror reflection of a 3D-object using continuous motion, the authors proposed controlled motion of stretched solids. This paper is devoted to distinguish novelty of such definition with assistance of artificial intelligence and to arrange such spaces in the theory of categories.

Keywords: kinematic space, artificial intelligence, axiomatization, controlled motion, motion of solids, computer implementation.
2020 MSC: 68U03

1. Introduction

The term "kinematic space" is used in various meanings. We mean "space with possiblity of controlled motion with bounded velocity." As regards a computer implementation: the distance between two screen images is defined as to the minimum transition time between them within the set of such images.

Remark. There is the Galileo-Einstein principle of relativity in real space in dual form: if a human watches uniform motion of an object, they will not be able to notice that either themself or the object is moving uniformly.

But in virtual spaces the restriction on uniform velocity is unnecessary and we obtain the computer principle of relativity (duality): If the user watches motion of an object on display then it can be interpreted either motion of the user or of the object.

The first interpretation is used in computer games and is considered in this paper and the second one is used in mathematical software such as Excel, MathCad.

As far as is known, S. Ulam [1] was the first to offer using a computer to treat spaces other than the three-dimensional Euclidean one. He proposed a problem that can be solved - passing a four-dimensional body through a hole with a two-dimensional boundary.

Anticipating future investigations and implementations, A. Rosenfeld [2] introduced the term "Digital topology".

Since the advent of personal computers, there have been miscellaneous attempts in this direction.

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Many programmers independently discovered the following technique: when an object disappears from a screen, it appears from opposite side of a screen, i.e. the object is in motion in a topological torus.

Remark. Some of programmers thought that they programmed a sphere (a bounded surface without boundaries).

A well-known trick: when a flat object "rotates" and each time a new image appears on its side, can be considered as a movement in the Riemann surface for the function $w = Ln(z)$.

The "door to another world" technique was also used many times. It can be considered as a motion in the Riemann surface for the function $w(z) = \sqrt{(}z^2 - a^2)$.

J.R. Weeks [3] proposed "to look through" along bounded non-Euclidean objects.

Videoclips demonstrating various topological transformations, rotations were made and are being made, for instance see [4].

E. Sandler [5] demonstrated videoclips including "life" on 2D-surfaces.

A.Borubaev, P. Pankov [6] proposed axiomatization of controlled motion of a point (avatar of the user) and gave a stict definition of a kinematical space. They implemented various 2D-spaces: Riemann surfaces, Moebius band, projective plane, topological torus, and 4D-Euclidean space.

Remark. Actually, the traveller not only "passes" but also "changes" the space. This is provided in [6]: there are poles, there is a target, the user breaks the poles when running into them (intentionally or accidentally).

Remark. The control was implemented both in the side-scroller mode and in the cockpit view mode. In the cockpit view mode controlled rotation of the direction of view was implemented. Thus, the 2D-space of points becomes a 3D-space of "points with horizontal directions".

This monograph has been reissued in an expanded form [7].

We proposed controlled motion of stretched solids. This paper is devoted to distinguish novelty of our definition with assistance if artificial intelligence and to arrange such spaces in the theory of categories.

2. Preliminaries and Definitions

2.1. Definitions

The most general category presenting the idea of continuity is *Top* of topological spaces. Its subcategory is the category *Unif* of uniform spaces, in turn, its subcategory is *Met* of metrical spaces. The category *Cntl* of controlled processes was proposed in [7].

As a codifying the ancient idea of controlled movement with bounded velocity and measuring "distance" by "time", the notion of kinematical space was introduced [2] and declared as subcategory *Kin* of *Met* and of *Cntl* in [6]:

Definition 1 (non-formal). A computer program is said to be a presentation of a computer kinematic space if:

P1) there is an (infinite) metrical space X of points and a set X_1 of display-presentable points being sufficiently dense in X;

P2) the user can pass from any point $x_1 \in X_1$ to any other point $x_2 \in X_1$ by a sequence of adjacent points in X_1 by their will;

P3) the minimal time to reach x_2 from x_1 is (approximately) equal of the minimal time to reach x_1 from x_2.

The minimal time to pass from x_1 to x_2 is the kinematic distance between these points, $\rho_{kin}(x_1, x_2)$.

As a consequence,

P4) $\rho_{kin}(x_1, x_3) \leq \rho_{kin}(x_1, x_2) + \rho_{kin}(x_2, x_3)$.

We [3], [4] proposed

Definition 2. There is a family P of connected sets of the topological space X (we will call them passes); each pass has the positive length (time) and a family Q of connected sets of X (we will call them things). (It means that a thing moves along a pass).

The space X is said to be a solid kinematic space if:

(G1) For each $x \in p \in P$ there exists such $q \in Q, q \subset p$ that $x \in q$ [a thing can be in each place of a pass].

(G2) For each pair $x_1 \neq x_2 \in X$ (such that there is not such $q \in Q$ that $x_1, x_2 \in q$) there exists such pass $p \in P$ that $x_1, x_2 \in p$ and the set of lengths of such p is bounded below with a positive number; this infimum is said to be the point-kinematical distance in X between these points within things.

(G3) For each homeomorphic pair $q_1 \neq q_2 \in Q$ there exists such pass $p \in P$ that $q_1, q_2 \subset p$ and the set of lengths of such p is bounded below with a positive number; this infimum is said to be the solid-kinematical distance in X between things q_1 and q_2.

(G4) If $x_1, x_2 \in p_1$ and $x_2, x_3 \in p_2$ then there exists such pass $p_3 \in P$ that $x_1, x_2, x_3 \in p_3$ and $length(p_3) \leq length(p_1) + length(p_2)$.

We propose to call the category of such spaces as a *Kin-Solid* category.

2.2. Definitions by artificial intelligence

We asked: Write axiomatization for controlled motion of solids in topological space.

chat.deepseek.com responded:

Below is an axiomatization for the controlled motion of solids in a topological space, intended for a qualitative (topological) rather than metric or smooth setting. The focus is on continuity, control, and solidness (no tearing, no self-intersection, preserving topology of the solid).

Primitives

X: a topological space (ambient space).

S: a compact, connected, regular closed subset of X (a solid).

'Regular closed' means $S = cl(int(S))$ — no isolated parts, no missing boundary.

T: time interval, typically $[0, 1] \subset R$ with standard topology.

A motion is a map $\phi : S \times T \to X$.

Axioms

Axiom 1. Continuity of motion

$\phi : S \times T \to X$ is continuous.

(Product topology on $S \times T$, given $S \subset X$ with subspace topology.)

This ensures no sudden jumps; the solid moves through intermediate positions continuously.

Axiom 2. Instantaneous solidness

For each fixed $t \in T$, the set $\phi_t(S) = \{\phi(x, t) : x \in S\}$ is homeomorphic to S. Equivalently: $\phi_t : S \to \phi_t(S)$ is a homeomorphism onto its image.

This prevents collisions with itself, tearing, or pinching — the solid remains topologically a solid (no identification of distinct points at any moment).

Axiom 3. Control axiom (openness of control space)

Let O be an open set in X containing $\phi_t(S)$ for some t. There exists $\varepsilon > 0$ such that for all $t' \in T, tt' < \varepsilon$, $\phi'_t(S) \subset O$.

Moreover, control action means: For any desired open neighborhood U of a target configuration, there exists a continuous path in the configuration space $Homeo(S, X)$ of topological

embeddings (with compact-open topology) from the current embedding to one whose image lies in U.

Formally: Let $Emb(S, X)$ be the space of topological embeddings of S into X. The motion ϕ induces a path $\phi^* : T \to Emb(S, X), \phi^*(t) = \phi_t$.

The control axiom: ϕ^* is continuous and locally controllable — for any $\phi^*(t_0)$ and any neighborhood N^* of $\phi^*(t_0)$ in $Emb(S, X)$, there exists $\delta > 0$ and a motion ψ such that $\psi_{t_0} = \phi^*(t_0)$ and $\psi_{t_0+\delta} \in N^*\}$ with trajectory inside N^* for $[t_0, t_0 + \delta]$.

Axiom 4. Non-self-intersection of trajectory For any two different times $t_1 \neq t_2$, it is not required that $\phi_{t_1}(S) \cap (S) = \varnothing$ — solids may pass through same region at different times (no non-intersection condition). However: No interpenetration at the same time is already given by Axiom 2 (homeomorphism).

An additional axiom: For any t, $\phi_t(S) \cap \phi_t(\partial S) = \phi_t(\partial S)$ trivially, but more importantly, for two distinct solids A, B if we had multiple solids, we'd require disjointness at each t; here single solid.

Axiom 5. Topological isotopy condition

The map $\Phi : S \times T \to X \times T, \Phi(x, t) = (\phi(x, t), t)$ is an embedding of $S \times T$ into $X \times T$.

Equivalently: The motion is an ambient isotopy of S in X restricted to S (can be extended to an isotopy of X). This ensures that the solid's topology is preserved over time and the motion is "nice" globally.

Axiom 6. Response to control input

Let C be the set of control parameters (e.g., a topological space). For each $c \in C$, there exists a motion $S \times T \to X$ satisfying Axioms 1–5.

A controlled motion is a continuous map $\Gamma : C \times T \to Emb(S, X), \Gamma(c, t) = \phi_{tc}$ such that $\Gamma(c, \cdot)$ satisfies Axioms 1–5 for each fixed c. The control action axiom: The evaluation map $C \to Emb(S, X)$ given by $c \mapsto \Gamma(c, 0)$ is continuous, and for any desired continuous path in $Emb(S, X)$ from an initial embedding e_0 to some target, there exists a path in C whose induced motion approximates it.

Interpretation.

This setup is qualitative: exact distances are absent; only continuity, openness, embedding, and isotopy remain. It models e.g., robotic manipulation where the solid cannot be torn or self-intersect, and control changes motion continuously. The topological space X could be R^n, but also a manifold or a more exotic space (e.g., digital topology, CW complex).

2.3. Analysis of artificial intelligence's respond

Artificial intelligence did not mention concatenation of paths with adding times (G4 in Defenition 2). All paths in its system of axioms has the same time $T^* = 1$ which does not correspond reality. Hence, Definition 2 is original. At the same time, it is convenient to represent a pass as a continuous function, as suggested by the artificial intelligence.

Continuation of Definition 2:

A pass is a triple:

an element $q \in Q$;

a number $T^* \in R_+$;

a continuous function $\phi : q \times [0, T^*] \to exp(X)$

such that $\phi(x, 0) = x$ and for any $t \in (0, T^*]$ the inverse function $\phi^{-1} : \phi(q, t) \to q$ exists and is continuous.

(G4') If there are passes $\phi_1 : q \times [0, T^*] \to X$ and $\phi_2 : q \times [0, T^{**}] \to X$ and the function $\phi_3 : q \times [0, T^* + T^{**}] \to X$ defined as

$\phi_3(q, t) = \phi_1(q, t)$ for $t \in [0, T^*]$,

$\phi_3(q,t) = \phi_2(q, t - T^*)$ for $t \in (T^*, T^{**}]$
is continuous then the triple $\{q, T^* + T^{**}, \phi_3\}$ is a pass.

3. Main Results

As Definition 2 is new, we may specify it:

Definition 3. If for any pass of a there exists a connected (non–empty) subset (axis) $C \subset q$ such that

(R1) $(\forall t \in [0, T^*])(\phi(x,t) = x)$(the axis does not move);

(R2) $(\forall x \in q)(\phi(x, T^*) = x)$ the set q makes a full turn);

(R3) $(\forall t_1 < t_2 \in (0, T^*))(\phi(q, t_1) \cap \phi(q, t_2) = C)$ (the set q is "one–layer" and does not pass by itself excluding the axis C);

then such pass is proper rotation of q around the axis C.

Unlike [4] we [9] implemented controlled rotations in 4D-Euclidean space with a concrete task for the user.

The user sees a 3D-solid (for example, a left boot) on the screen which is a 3D-slit (W=0) to 4D-space with coordinates X, Y, Z, W. The user may choose one of pairs XY, XZ, XW, YZ, YW, ZW. Then they see the projection of the solid on the 2D-space (for example, YW) and can rotate the solid around the axis (XZ in this example). The user is to obtain the mirror image (to change orientation) of the solid on the screen (for example, to obtain a right boot).

The well-known fact that rotation of any 3D-object in R^4 can change its orientation in R^3 was illustrated.

4. Conclusion

We hope that controlled motion of solids in virtual spaces, as well as other controlled transformations would promote deep learning of mathematics and physics and independent computer presentation of natural languages (see [10] for instance).

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Singularly Perturbed Equations with Poles of Different Multiplicities

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Abstract

This paper considers a system of singularly perturbed equations consisting of four first-order equations in a certain domain of the complex plane. The matrix function of the leading approximation has four distinct eigenvalues forming two pairs of complex conjugates. One pair has simple poles, while the other has double poles. The problem of establishing the validity of the limiting transition in a certain subdomain of the region under consideration is formulated. This problem is solved using the method of level lines and successive approximations.

Keywords: singular perturbation, degenerate equation, pole, level lines, asymptotic estimates, successive approximations, limiting transition.
2020 MSC: 34E05.

1. Introduction

Some formulas are presented in a form convenient for automatic and artificial-intelligence-based symbolic processing.

Consider the equation

$$\varepsilon x'(t) = \Lambda(t)x(t) + \varepsilon b(t) + f(t, x(t)), \tag{1.1}$$

with the initial condition

$$x(t_0, \varepsilon) = x^0, \tag{1.2}$$

where $0 < \varepsilon$ is a small real parameter;

$t \in D \subset \mathbb{C}$, where D is an open domain and $t_0 \in D$; $\Lambda(t) = \mathrm{diag}(\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t))$, $b(t) = (b_1(t), b_2(t), b_3(t), b_4(t))$, $\quad x = (x_1, x_2, x_3, x_4)$, $\quad f = (f_1, f_2, f_3, f_4)$.

Equations of type (1.1) are usually referred to as "singularly perturbed equations" (SPE) [13]. The main results for SPE in the case where (t) is a real variable are presented in [10, 8, 13, 14].

A central problem in the theory of singular perturbations is the problem of the limiting transition. Its essence is as follows. Setting $\varepsilon = 0$ in (1.1), we obtain the so-called "degenerate system"

$$\Lambda(t)\xi + f(t, \xi) = 0. \tag{1.3}$$

Assume that system (1.3) has a solution ($\xi = \xi_0(t)$), defined for ($t \in [t_0, T]$), a segment of the real axis. Suppose that, under certain conditions, there exists a solution ($x(t, \varepsilon)$) of problem (1.1)–(1.2), also defined on ($t \in [t_0, T]$).

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Problem. The main question is whether the following relation holds:

$$\lim\{x(t,\varepsilon) : \varepsilon \to 0,\ t \in [t_0, T]\} = \xi_0(t). \tag{1.4}$$

In its most general form, this problem for a wide class of singularly perturbed equations has been solved in [10, 13]. Various related problems for singular perturbations have also been studied in [5, 6, 7, 12, 9].

2. Preliminaries

Assume that the following conditions hold:

(A1) The functions $\lambda_j(t), j = 1, 2, 3, 4$, and $b_j(t)$ belong to the space $Q(H)$ of functions analytic in the domain D.

(A2) The function $f(t,x) \in Q(H)$, where $H = \{(t,x) : t \in D, \|x\| \le C_1\}$, and $\|x\| = \max\{|x_j|\}, \quad f(t,0) \equiv 0$.

(A3) $\forall\big((t,\tilde{x}),(t,\tilde{\tilde{x}})\big) \in H : \quad \|f(t,\tilde{x}) - f(t,\tilde{\tilde{x}})\| \le C_2\|\tilde{x} - \tilde{\tilde{x}}\| \cdot \max\left\{\|\tilde{x}\|, \|\tilde{\tilde{x}}\|\right\}^2$.

We restrict our attention to the case where $\lambda_{1,2}(t) = \frac{2}{t\pm i}, \qquad \lambda_{3,4}(t) = \frac{-2}{(t\pm i)^2}$.

The functions $\lambda_{1,2}(t)$ have simple poles at the points $t = \pm i$, whereas $\lambda_{3,4}(t)$ have double poles at the same points.

In what follows, we assume that $D = \{t \in \mathbb{C} : |t| < r_0\}$, where r_0 is a sufficiently large positive real number.

The degenerate system corresponding to (1.1), in view of assumption (A2), admits the trivial solution $\xi(t) \equiv 0$.

3. Main results

Under the above assumptions, there exist a subdomain $D_0 \subset D$ and a solution $x(t,\varepsilon)$ of problem (1.1)–(1.2), defined in D_0, such that

$$\lim\{x(t,\varepsilon) : \varepsilon \to 0\} = 0, \qquad \forall t \in D_0. \tag{3.1}$$

A distinctive feature of the problem under consideration is that, along the real axis, the real parts of the eigenvalues $\lambda_{1,2}(t)$ change sign at $t = 0$, namely:

$\operatorname{Re}\lambda_{1,2}(t) < 0 \quad$ for $t < 0, \qquad \operatorname{Re}\lambda_{1,2}(t) > 0 \quad$ for $t > 0$.

At the same time, for $\lambda_{3,4}(t)$, we have:

$\operatorname{Re}\lambda_{3,4}(t) < 0 \quad$ for $t < -1$,

$\operatorname{Re}\lambda_{3,4}(t) > 0 \quad$ for $-1 < t < 1$,

$\operatorname{Re}\lambda_{3,4}(t) < 0 \quad$ for $t > 1$.

Thus, it is necessary to prove the existence of a domain D_0, containing a segment of the real axis, in which at least one pair of eigenvalues changes sign from negative to positive.

We transform problem (1.1)–(1.2) into the equivalent integral equation

$$x(t,\varepsilon) = x^0 E(t_0,t,\varepsilon) + \frac{1}{\varepsilon}\int_{t_0}^{t} E(\tau,t,\varepsilon)\big[\varepsilon b(\tau) + f(\tau, x(\tau,\varepsilon))\big]d\tau \tag{3.2}$$

where $E(\tau,t,\varepsilon) = \operatorname{diag}\left(\exp\left\{\frac{1}{\varepsilon}\big(F_1(t) - F_1(\tau)\big)\right\}, \ldots, \exp\left\{\frac{1}{\varepsilon}\big(F_4(t) - F_4(\tau)\big)\right\}\right)$.

The functions $F_j(t)$ are given by $F_{1,2}(t) = 2\ln(t \pm i), \quad F_{3,4}(t) = \frac{2}{t\pm i}$.

Choose $t_0 < -1$. Then, without loss of generality, one may assume that

$$\|x^0\| \ \le C_3\varepsilon. \tag{3.3}$$

Applying the method of successive approximations to equation (3.2), define the sequence

$$\begin{aligned} &x_m(t,\varepsilon) = x^0 E(t_0,t,\varepsilon) + \frac{1}{\varepsilon}\int_{t_0}^{t} E(\tau,t,\varepsilon)\left[\varepsilon b(\tau) + f(\tau, x_{m-1}(\tau,\varepsilon))\right] d\tau, m = 1,2,\ldots \\ &x_0(t,\varepsilon) \equiv 0, \end{aligned} \tag{3.4}$$

where $x_m = (x_{1m}, x_{2m}, x_{3m}, x_{4m})$.

At this stage, the following problems must be addressed:

1. To determine a suitable domain D_0.
2. To choose optimal integration paths ensuring boundedness of successive approximations.
3. To obtain estimates for successive approximations in D.
4. To prove uniform convergence of the successive approximations in D_0 to a function that solves the initial value problem.
5. To estimate the obtained solution in D_0 and thereby solve the stated problem.

The domain D_0 will be determined using the method of level curves ([1, 11, 2, 3, 4]). For this purpose, consider the functions $F_j(t)$, $j = 1,2,3,4$, and compute their real parts. Let $t = t_1 + it_2$, where $t_1, t_2 \in \mathbb{R}$. Then $\operatorname{Re} F_{1,2}(t) = \ln\left(t_1^2 + (t_2 \pm 1)^2\right)$, $\operatorname{Re} F_{3,4}(t) = \frac{2t_1}{t_1^2+(t_2\pm1)^2}$.

Consider, for instance, $\operatorname{Re} F_1(t)$ and $\operatorname{Re} F_3(t)$, and define their level curves:

$(p) = \left\{t \in D, \ln\left(t_1^2 + (t_2+1)^2\right) = p - \text{const}\right\}$, $(q) = \left\{t \in D, \ \frac{2t_1}{t_1^2+(t_2\pm1)^2} = q - \text{const}\right\}$.

The level curves of $\operatorname{Re} F_1(t)$ are concentric circles centered at the point $(0,-1)$ with radius $r = e^p$, (see Fig. 1).

Next, we determine the level curves of $\operatorname{Re} F_3(t)$. We obtain $\left(t_1 - \frac{1}{q}\right)^2 + (t_2+1)^2 = \frac{1}{q^2}$.

Thus, for $(q \neq 0)$, the level curves of $\operatorname{Re} F_3(t)$ are circles centered at the point $\left(\frac{1}{q}, -1\right)$ with radius $r_2 = \frac{1}{|q|}$.

These circles pass through the point $(0,-1)$; therefore, they can be regarded as punctured at this point, and their arcs are adjacent to it. In the case $q = 0$, the corresponding level curve is the imaginary axis of the complex plane (see Figure 1).

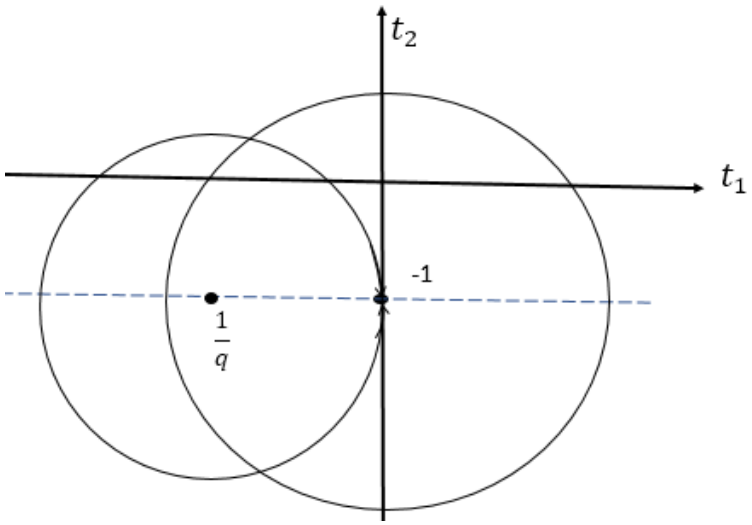


Figure 1. Level curves (p_0) and (q_0)

We now determine the level curves (p_0) and (q_0) passing through the point $(t_0, 0)$. Setting $t_1 = t_0, t_2 = 0$, we obtain $t_0^2 + 1 = r_1^2, \qquad p_0 = \ln(t_0^2+1)$. Further, $\left(t_0 - \frac{1}{q_0}\right)^2 + 1 = \frac{1}{q_0^2}$, which yields $t_0^2 - 2t_0\frac{1}{q_0} + 1 = 0$, and hence $q_0 = \frac{2t_0}{t_0^2+1}$.

Thus, the corresponding level curves are given by

$(p_0) = \left\{t \in D, \ln\left(t_1^2 + (t_2+1)^2\right) = \ln(t_0^2+1) \equiv p_0\right\}$,

$(q_0) = \left\{t \in D : \left(t_1 - \frac{1}{q_0}\right)^2 + (t_2+1)^2 = \frac{1}{q_0^2}\right\}$.

Next, we determine the intersection points of these level curves.

We have $\left(t_1 - \frac{1}{q_0}\right)^2 - t_1^2 = \frac{1}{q_0^2} - (t_0^2+1)$. From this, it follows that $t_1 \cdot \frac{t_0^2+1}{t_0} = t_0^2 + 1$, and hence $t_1 = t_0$.

Therefore, the level curves (p_0) and (q_0) intersect only at the point $(t_0, 0)$ and have no other common points.

To verify this, it suffices to compare their radii: $r_2^2 = \frac{1}{q_0^2}$.

It is easy to check that, under the condition $(t_0 < -1)$, $r_1^2 > r_2^2$.

Consequently, the level curve (p_0) encloses the level curve (q_0) (see Figure 2).

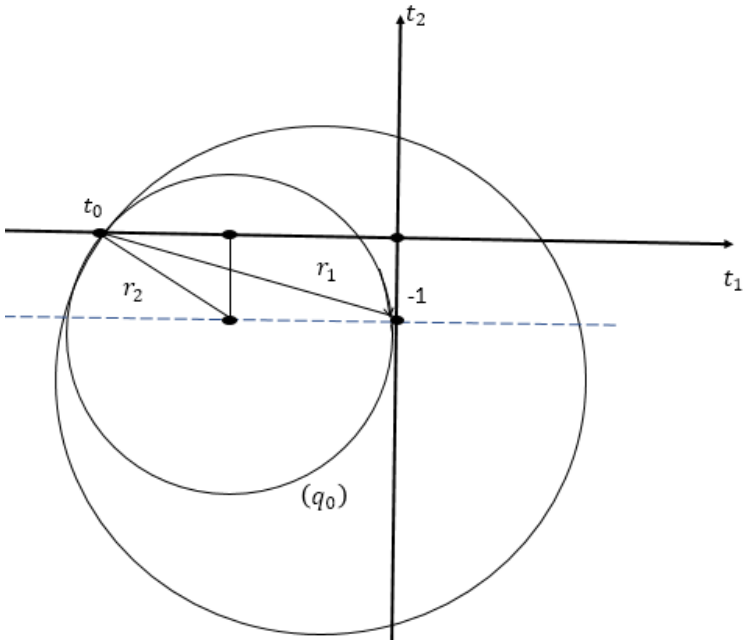


Figure 2. Level curves (p_0) and (q_0)

Next, we determine the intersection points of the curve (q_0) with the real axis. We have $\left(t_1 - \frac{1}{q_0}\right)^2 + 1 = \frac{1}{q_0^2}$, which reduces to $t_1^2 - \frac{2}{q_0}t_1 + 1 = 0$.

Solving this equation, we obtain the intersection points $A_1(t_0, 0), \quad A_2\left(\frac{1}{t_0}, 0\right)$.

Under the assumption $t_0 < -1$, it follows that $-1 < \frac{1}{t_0} < 0$.

Denote by (K) the arc of the curve (q_0) connecting the points A_1 and A_2.

Now consider the function $\operatorname{Re} F_4(t) = \frac{2t_1}{t_1^2+(t_2-1)^2}$.

The functions $\operatorname{Re} F_3(t)$ and $\operatorname{Re} F_4(t)$ take equal values at points symmetric with respect to the real axis. In what follows, symmetry will always be understood in this sense.

The function $\operatorname{Re} F_4(t)$ determines a curve $(\overline{K})$, symmetric to (K). The curve $(\overline{K})$ also connects the points A_1 and A_2.

Denote by D_0 the domain bounded by the curves (K) and $\overline{K}$.

In selecting the integration paths, we again employ the method of level curves ([1, 11, 2, 3, 4]). The principal criterion for choosing such paths is that, along them, the functions $\operatorname{Re} F_j(t)$ must not increase; that is, they must either remain constant or decrease.

According to this criterion, an admissible integration path should consist of segments of level curves together with curves that intersect each level curve at most once and in the direction of decreasing values.

Consider the curve (K). Along (K), the function $\operatorname{Re} F_3(t)$ remains constant, although it is not identically constant in the domain. Let us analyze the behavior of $\operatorname{Re} F_1(t)$ along (K).

From the equation of (K), we obtain $(t_2+1)^2 = -\left(t_1 - \frac{1}{q_0}\right)^2 + \frac{1}{q_0^2}$.

Since $\operatorname{Re} F_{1,2}(t) = \ln\left(t_1^2 + (t_2+1)^2\right)$, it follows that along (K),

$\operatorname{Re} F_1(t) = \ln\left(t_1^2 - \left(t_1 - \frac{1}{q_0}\right)^2 + \frac{1}{q_0^2}\right)$.

Differentiating along the curve (K), we obtain

$$\frac{d}{dt_1}\operatorname{Re} F_1(t) = \frac{\frac{2}{q_0}}{t_1^2 - \left(t_1 - \frac{1}{q_0}\right)^2 + \frac{1}{q_0^2}} < 0, \quad (t_0 < -1).$$

Hence, $\operatorname{Re} F_1(t)$ is strictly decreasing along the curve (K).

Next, consider the straight-line segment $(p_1) = \left\{\tau \in D_0 : \tau_1 = t_1, \quad t_2 \le \tau_2 \le \widetilde{t}_2\right\}$.

The segment (p_1) connects the points $\widetilde{t} = t_1 + i\widetilde{t}_2 \in (K)$, $\quad t = t_1 + it_2 \in D_0$. The motion along (p_1) is directed opposite to the t_2-axis (i.e., the imaginary axis).

We now examine the behavior of the functions $\operatorname{Re} F_1(t)$ and $\operatorname{Re} F_3(t)$ along (p_1).

For $\operatorname{Re} F_1(t)$, we have $\operatorname{Re} F_1(t) = \ln\left(t_1^2 + (\tau_2+1)^2\right)$ hence $\frac{\partial}{\partial \tau_2}\operatorname{Re} F_1(t) = \frac{2(\tau_2+1)}{t_1^2+(\tau_2+1)^2}$. Since for all $\tau \in D_0$ we have $(\tau_2+1) > 0$, it follows that along the direction opposite to the t_2-axis, $\frac{\partial}{\partial \tau_2}\operatorname{Re} F_1(t) < 0$.

Thus, along (p_1), in the chosen direction, the function $\operatorname{Re} F_1(t)$ is decreasing.

Next, for $\operatorname{Re} F_3(t)$, we obtain

$$\frac{\partial}{\partial \tau_2}\operatorname{Re} F_3(t) = \frac{\partial}{\partial \tau_2}\left(\frac{2t_1}{t_1^2 + (\tau_2+1)^2}\right) = \frac{-4t_1(\tau_2+1)}{\left(t_1^2 + (\tau_2+1)^2\right)^2}.$$

Under the conditions $t_1 < 0$ and $(\tau_2 + 1 > 0)$, it follows that $\frac{\partial}{\partial \tau_2}\operatorname{Re} F_3(t) > 0$ in the direction of the t_2-axis; hence, in the opposite direction it decreases.

Therefore, the integration paths can now be specified.

For the components x_{1m} and x_{3m}, the path consists of: the arc of (K) connecting the points $(t_0, 0)$ and $\widetilde{t} \in (K)$, and the straight-line segment connecting $\widetilde{t}$ with $t \in D_0$.

For the components x_{2m} and x_{4m}, the paths are chosen symmetrically with respect to those selected for x_{1m} and x_{3m}.

We derive estimates for the successive approximations using the method of mathematical induction.

Assume that the following estimate holds:

$$\|x_m(t,\varepsilon)\| \le C_4\alpha_m(\varepsilon), \qquad t \in D_0, \tag{3.5}$$

where $\alpha_m(\varepsilon)$ is a positive function of ε.

For $m = 1$, we have $\|x_1(t,\varepsilon)\| \le C_4\alpha_1(\varepsilon)$.

Let us determine $\alpha_1(\varepsilon)$. According to the chosen integration paths,

$\operatorname{Re} F_j(t) - \operatorname{Re} F_j(t_0) \le 0$, $\quad \operatorname{Re} F_j(t) - \operatorname{Re} F_j(\tau) \le 0$. Moreover, the domain D_0 does not contain singular points of the functions involved.

Thus, for the first approximation, we have
$x_1 = x^0 E(t_0, t, \varepsilon) + \int_{t_0}^{t} E(\tau, t, \varepsilon)\, b(\tau)\, d\tau$.
Since $\|E(t_0, t, \varepsilon)\| \leq 1, \quad \|x^0\| \leq C_3\varepsilon$, it follows that $\|x_1(t, \varepsilon)\| \leq C_4\varepsilon$.
Therefore, $\alpha_1(\varepsilon) = \varepsilon$.
Next, we proceed with the estimation. We obtain

$$\begin{aligned}\|x_{m+1}(t, \varepsilon)\| &\leq \|x_1(t, \varepsilon)\| + \frac{1}{\varepsilon}\left|\int_{t_0}^{t} \|E(\tau, t, \varepsilon)\|\, \|f(\tau, x_m(\tau, \varepsilon))\|\, d\tau\right| \\ &\leq \|x_1(t, \varepsilon)\| + \frac{C_2}{\varepsilon}\left|\int_{t_0}^{t} \|x_m(\tau, \varepsilon)\|^3 \|E(\tau, t, \varepsilon)\|\, d\tau\right|.\end{aligned}$$

Therefore,

$$\|x_{m+1}(t, \varepsilon)\| \leq C_4\varepsilon + \frac{C_0 C_2 C_4^3}{\varepsilon}\alpha_m^3(\varepsilon).$$

This yields the recurrence relation

$$\alpha_{m+1}(\varepsilon) = \varepsilon + \frac{C_0 C_2 C_4^2}{\varepsilon}\alpha_m^3(\varepsilon), \quad m = 1, 2, \ldots$$

For $m = 1$, we obtain

$$\alpha_2(\varepsilon) = \varepsilon + \frac{C_0 C_2 C_4^2}{\varepsilon}\alpha_1^3(\varepsilon) = \varepsilon + C_0 C_2 C_4^2 \varepsilon^2 = \varepsilon\left(1 + C_0 C_2 C_4^2, \varepsilon\right).$$

Assume that $1 + C_0 C_2 C_4^2 \varepsilon \leq C_5$ that is $\varepsilon \leq \frac{C_5 - 1}{C_0 C_2 C_4^2}, \qquad (C_5 > 1)$.
This condition is satisfied for sufficiently small ε. Hence, $\alpha_2(\varepsilon) \leq C_5\varepsilon$.
For $m = 2$, we obtain

$$\alpha_3(\varepsilon) = \varepsilon + \frac{C_0 C_2 C_4^2}{\varepsilon} C_5^3 \varepsilon^3 = \varepsilon + C_0 C_2 C_4^2 C_5^3 \varepsilon^2 = \varepsilon\left(1 + C_0 C_2 C_4^2 C_5^3 \varepsilon\right).$$

Assuming that $1 + C_0 C_2 C_4^2 C_5^3 \varepsilon \leq C_5$, we obtain $\alpha_3(\varepsilon) \leq C_5\varepsilon$.
Proceeding inductively, we arrive at $\alpha_{m+1}(\varepsilon) \leq C_5\varepsilon, \qquad m = 0, 1, 2, \ldots$
Thus,

$$\|x_{m+1}(t, \varepsilon)\| \leq C_4 C_5 \varepsilon = C_6\varepsilon, \quad C_6 > C_4, \quad m = 0, 1, 2, \ldots, \quad t \in D_0. \tag{3.6}$$

To establish uniform convergence, we apply the Weierstrass convergence test.
We estimate the differences $\|x_m - x_{m-1}\|,\ m = 1, 2, \ldots$
For $m = 1$,

$$\|x_1(t, \varepsilon)\| \leq C_6\varepsilon.$$

For $m = 2$, using assumption $(A3)$, we obtain

$$\|x_2 - x_1\| \leq \frac{1}{\varepsilon}\left|\int_{t_0}^{t} C_2 \|x_1(\tau, \varepsilon)\|^3 \|E(\tau, t, \varepsilon)\|\, d\tau\right| \leq C_2 C_0 C_6^3 \varepsilon^2 = C_6\varepsilon\left(C_0 C_2 C_6^2 \varepsilon\right).$$

Similarly,

$$\|x_3 - x_2\| \leq \frac{1}{\varepsilon}\left|\int_{t_0}^{t} C_2 \|x_2 - x_1\| \max\{\|x_1\|, \|x_2\|\}^2 \|E(\tau, t, \varepsilon)\|\, d\tau\right| \leq C_6\varepsilon\left(C_0 C_2 C_6^2 \varepsilon\right)^2.$$

Hence,

$$\|x_3 - x_2\| \le C_6\varepsilon \left(C_0C_2C_6^2\varepsilon\right)^2.$$

Proceeding similarly,

$$\|x_4 - x_3\| \le \frac{1}{\varepsilon}\left|\int_{t_0}^{t} C_2\|x_3 - x_2\| \max\{\|x_2\|, \|x_3\|\}^2 \|E(\tau, t, \varepsilon)\|\, d\tau\right| \le C_6\varepsilon \left(C_0C_2C_6^2\varepsilon\right)^3.$$

In general, we obtain the estimate

$$\|x_m - x_{m-1}\| \le C_6\varepsilon \left(C_0C_2C_6^2\varepsilon\right)^{m-1}, \quad m = 1, 2, \ldots$$

Hence, under the condition $C_0C_2C_6^2\varepsilon < 1$, it follows that the sequence of successive approximations (3.4) converges uniformly in D_0 to a function $x(t, \varepsilon)$, which is a solution of equation (3.2).

According to estimate (3.6), for the solution $x(t, \varepsilon)$ we obtain

$$\|x(t, \varepsilon)\| \le C_6\varepsilon, \qquad \forall t \in D_0. \tag{3.7}$$

Consider problem (1.1)–(1.2) under assumptions (A1)–(A3), and suppose that $\lambda_{1,2}(t) = \frac{2}{t\pm i}$, $\lambda_{3,4}(t) = \frac{-2}{(t\pm i)^2}$, with the initial condition satisfying $\|x(t_0, \varepsilon)\| \le C_3\varepsilon$.

Then there exists a domain $D_0 \subset D$ and a solution $x(t, \varepsilon)$ of problem (1.1)–(1.2), defined in D_0, for which estimate (3.7) holds.

Estimate (3.7) implies the validity of relation (3.1). Thus, the problem of the limiting transition is resolved.

The domain D_0 contains the segment $[t_0, \frac{1}{t_0}]$, $\quad (t_0 < -1)$, in which $\operatorname{Re}\lambda_{3,4}(t)$ change sign from negative to positive. Hence, the problem is completely solved.

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Computer Software for Constructing a 3D Model of the Spatial Geometric Fractal "Limit Cube of Cubes"

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Abstract

This paper presents a computer program developed for constructing a 3D model of the geometric fractal "Limit Cube of Cubes". The objects of study are planar and spatial geometric fractals. To generate the 3D model of the spatial geometric fractal "Limit Cube of Cubes", the Iterated Function System (IFS) method was employed. As a result, a 3D model of the fractal was obtained, and it was demonstrated that this fractal possesses a "limit surface" in the form of a cube. The novelty of the obtained result lies in the construction of a new spatial fractal with a cubic limit surface. The proposed fractal may find applications not only in geometry but also in various fields of physics and engineering. The obtained results can be used in further research on fractal theory and its applications.

Keywords: planar geometric fractal, spatial geometric fractal, 3D model, cube, limit surface, Iterated Function System (IFS), programming language, computer software.
2020 MSC: 28A80

1. Introduction

The concept of "fractal" was introduced by Benoit Mandelbrot in 1975. In his book "Fractal Geometry of Nature", he writes that mathematicians of the past years have always refused to study the forms that nature shows us, studying Euclidean geometric shapes and inventing all possible theories that do not explain the reality around us. However, according to Mandelbrot, "...The new geometry is able to describe many of the irregular and fragmented shapes in the world around us and generate complete theories, defining a family of shapes that I call fractals" [3].

From a mathematical point of view, a fractal is primarily a set of fractional dimensions. Anyone who studies geometry knows that the dimension of a segment is 1, a square is 2, a cube and a parallelepiped are 3. Fractional dimension is the main property of fractals.

The birth of fractal geometry is usually associated with the publication of Mandelbrot's book "Fractal Geometry of Nature" in 1977. It uses the scientific results of scientists who worked in the same field in the period 1875-1925. Among them are Poincare, Fatou, Julia, Kantor, Hausdorff. Only in our time has it been possible to combine these works into a single system [7].

Fractal geometry is a revolution in mathematics and mathematical description of nature. This is how the discoverer of fractal geometry, B. Mandelbrot, writes about it: "Why is geometry often called cold and dry? One of the reasons is her inability to describe the shape of a cloud, mountain, tree, or seashore.

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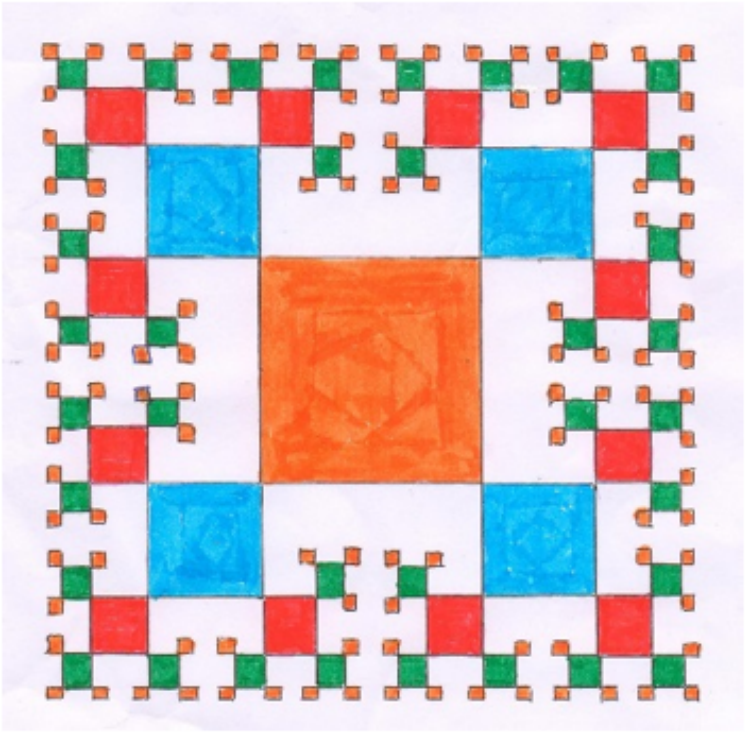
Clouds are not spheres, mountains are not cones, coastlines are not circles, and the crust is not smooth, and lightning does not travel in a straight line... Nature shows us not just a higher degree, but a completely different level of complexity. The number of different length scales in structures is always infinite."

A new figure, a fractal, can act as a model of complex natural systems, such as tree crowns, mountain ranges, coastlines, the surface of the moon, etc. Tree fractals are used to model not only plants, but also the bronchial tree, kidneys, and circulatory system.

In the article [6], a computer program was developed and a 3D model of the spatial fractal "Sphere with six towers" was constructed, the "limit surface" of which is an octahedron. In the article [5], a 3D model of the spatial fractal "Sphere with fourteen towers" is constructed. In this article, a computer program has been developed for constructing a 3D model of a geometric fractal "Sphere with twenty towers", the "limiting surface" of which is a dedocahedron.

2. Research materials

This article discusses the problem of constructing a 3D model of a spatial geometric fractal [4] (1-Figure) "Limit cube of cubes" based on the well-known flat fractal "Limit square of squares". It is assumed that this planar fractal is an orthogonal projection of some spatial self-similar object with cubic symmetry and having a limiting surface in the shape of a cube.



1-drawing. The ultimate square of squares.

To solve this problem, the methods of computer modeling and the theory of iterative processes are used. The main tool for constructing a three-dimensional structure is the Iterated Function System (IFS), which makes it possible to form complex self-similar geometric objects through sequential scaling and transformation of basic elements.

As part of the study, the following task was set: to reconstruct its spatial counterpart from a given flat fractal and obtain an interactive 3D model of the "Limit Cube of Cubes"

fractal, as well as to investigate its limiting behavior with an increase in the number of iterations.

Several basic approaches to generating fractal structures are used to build a spatial model.:

- Recursive Algorithms;
- Iterated Function System (IFS);
- Multidimensional graphical visualization (Multidimensional Graphics);
- L-systems (Lindenmayer systems).

In this paper, the main attention is paid to the IFS method, as the most versatile and convenient for software implementation in three-dimensional modeling tasks.

The geometric structure of the "Limit Cube of Cubes" fractal is formed by dividing the original cube into smaller similar elements located at the nodal points of the cubic lattice. The coordinates of these points are set using a vector system reflecting the symmetry of the cube and its diagonal directions. In the simplest case, the vertices of the unit cube are used:

$$\vartheta^1(1,1,1), \vartheta^2(1,1,-1), \vartheta^3(1,-1,1), \vartheta^4(1,-1,-1),$$
$$\vartheta_5(-1,1,1), \vartheta_6(-1,1,-1), \vartheta_7(-1,-1,1), \vartheta_8(-1,-1,-1).$$

Next, an iterative transformation is introduced in which each element of the structure is replaced by a set of scaled-down copies of the original object with a scaling factor r and corresponding shift vectors t_i:

$$S_{n+1} = \cup f_i(S_n), \quad i = 1, 2, \ldots, N,$$

where f_i are affine transformations of the form:

$$f_i(x, y, z) = r \cdot (x, y, z) + t_i.$$

This transformation ensures self-similarity of the structure and the formation of a multi-level fractal composition, in which each subsequent level is a scaled-down copy of the previous one.

The fractal dimension of the object under study is determined by the Hausdorff formula:

$$\frac{\log(N)}{\log(1/r)}$$

where N is the number of self–similar elements at each iteration step, and r is the scaling factor.

Vector transformations of the displacement t_i are set based on the vertices of the cube and ensure a uniform distribution of elements in space, while maintaining the cubic symmetry of the structure.

Thus, the mathematical model used allows us to consistently build a spatial fractal "Limit cube of cubes" with a given number of iteration levels, providing its visualization in the form of an interactive 3D scene and the study of its ultimate geometric shape.

The resulting model demonstrates the property of self-similarity and confirms the hypothesis of the existence of a limiting surface corresponding to a cube with an unlimited increase in the number of iterations.

Listing 1 – 3D Fractal construction Program

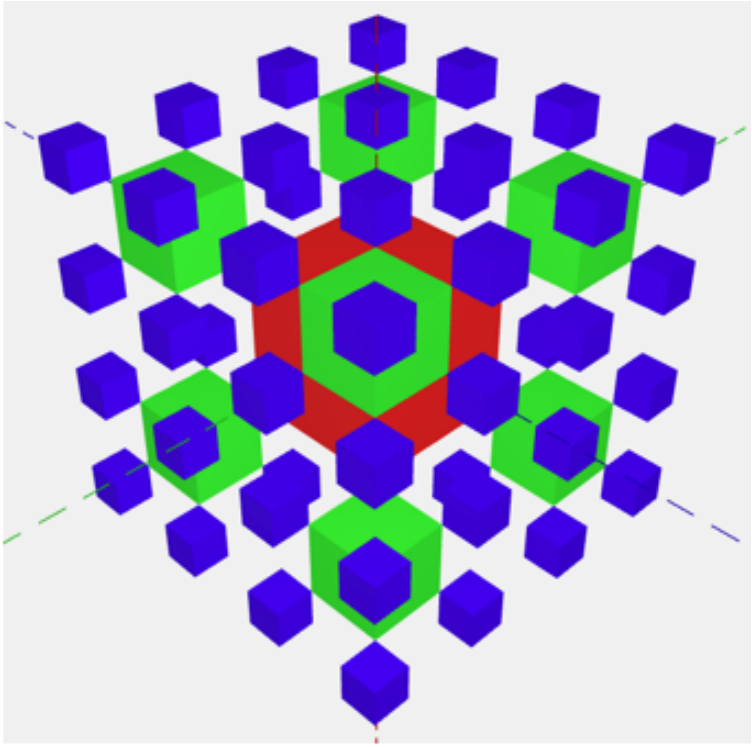
```
// Fractal Parameters
let lLen = 4;          // number of iteration levels
let fr = 1;            // initial cube size
let rlKaf = 0.5;       // Zoom level
let cubecalc = [];     // calculation nodes of a fractal
let cubes = [];        // 3D objects
// GUI level management
class inputGUIa {
    constructor() {
        this.group_1 = true;
        this.group_2 = true;
        this.group_3 = true;
        this.group_4 = false;
    }
}
let inputGUI = new inputGUIa();
let gui = new dat.GUI({ closeOnTop: true, width: 250 });
let checkLv = [
    gui.add(inputGUI, 'group_1').name(' Level 1 (1)'),
    gui.add(inputGUI, 'group_2').name(' Level 2 (21)'),
    gui.add(inputGUI, 'group_3').name(' Level 3 (421)'),
    gui.add(inputGUI, 'group_4').name(' Level 4 (8421)')
];
// The colors of the levels
let sColor = [
    { r: 1, g: 0.1, b: 0.1 },
    { r: 0.1, g: 1, b: 0.1 },
    { r: 0.2, g: 0.2, b: 1 },
    { r: 1, g: 0.8, b: 0.1 },
    { r: 0.2, g: 1, b: 1 },
    { r: 1, g: 0.2, b: 0.7 },
    { r: 1, g: 0.1, b: 1 }
];
// Vectors of cubic symmetry
class CubeVectors {
    constructor() {
        this.vs = [
            new THREE.Vector3( 1,  1,  1),
            new THREE.Vector3( 1,  1, -1),
            new THREE.Vector3( 1, -1,  1),
            new THREE.Vector3( 1, -1, -1),
            new THREE.Vector3(-1,  1,  1),
            new THREE.Vector3(-1,  1, -1),
            new THREE.Vector3(-1, -1,  1),
            new THREE.Vector3(-1, -1, -1)
        ];
        return this.vs.map(v => v.normalize());
    }
```

```
}
// Recursive fractal construction
class Cubecalc {
    constructor(pos, r, lv) {
        this.pos = pos;
        this.r = r;
        this.lv = lv;
        if (this.lv < lLen - 1) {
            let dv = new CubeVectors();
            dv.forEach(el => {
                let nPos = new THREE.Vector3();
                let dist = this.r * (rlKaf + 0.99) / 1.15;
                cubecalc.push(
                    new Cubecalc(
                        nPos.addVectors(this.pos,
                        el.multiplyScalar(dist)),
                        this.r * rlKaf,
                        this.lv + 1
                    )
                );
            });
        }
    }
}
// Initializing a fractal
cubecalc.push(new Cubecalc(new THREE.Vector3(), fr, 0));
// Geometry and material
let dgeometry = new THREE.BoxGeometry(fr, fr, fr);
let dmaterial = new THREE.MeshPhongMaterial();
let oMesh = new THREE.Mesh(dgeometry, dmaterial);
// Creating an array of objects
cubeArr();
cubesUpdate();
function cubeArr() {
    cubes = [];
    cubecalc.forEach(el => {
        let meshNew = oMesh.clone();
        meshNew.material = oMesh.material.clone();
        meshNew.material.color = sColor[el.lv];
        cubes.push(meshNew);
    });
}
// Updating the scene
function cubesUpdate() {
    let groupS = new THREE.Group();
    scene.traverse(child => {
        if (child.name === "asd") {
            scene.remove(child);
```

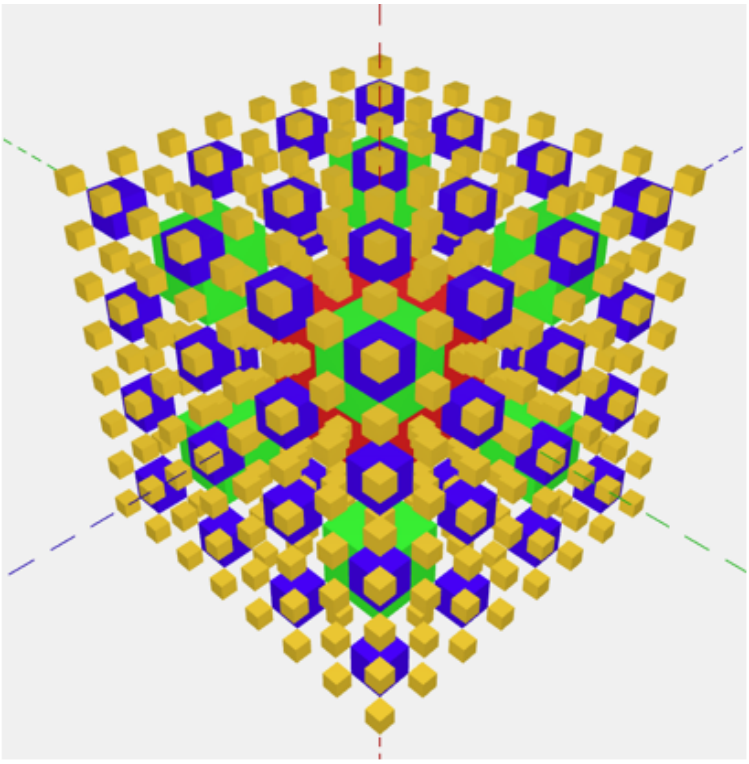
```
        }
    });
    cubes.forEach((el, i) => {
        el.position.copy(cubecalc[i].pos);
        let kf = cubecalc[i].r / fr;
        el.scale.set(kf, kf, kf);
        groupS.add(el);
    });
    groupS.name = "asd";
    scene.add(groupS);
    lvCheck(false, 3);
}

// Level management
function lvCheck(v, l) {
    cubes.forEach((el, i) => {
        if (cubecalc[i].lv === l) {
            el.visible = v;
        }
    });
}
checkLv.forEach((el, i) => {
    el.onChange(v => lvCheck(v, i));
});
// Rendering and looping
function update() {}
function render() {
    renderer.render(scene, camera);
}

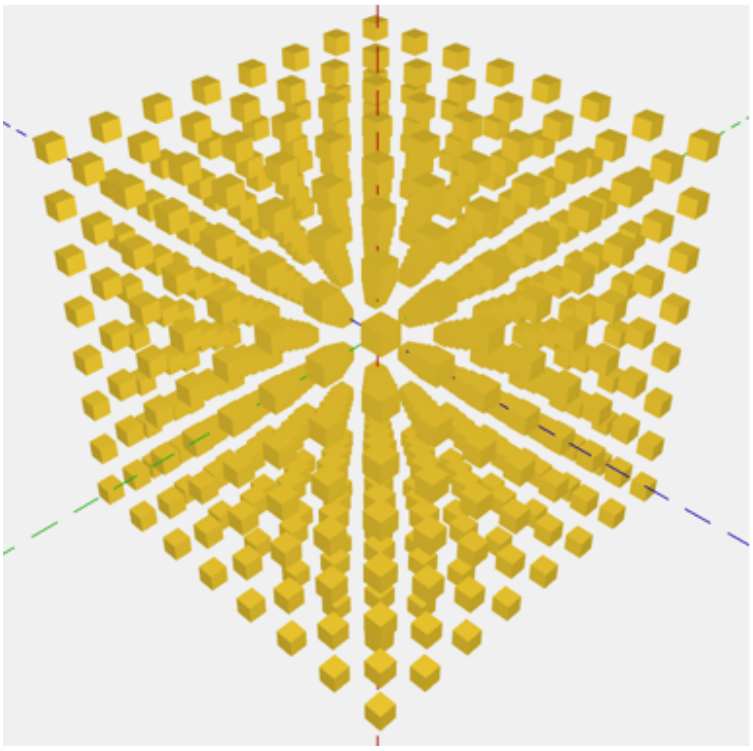
function Gameloop() {
    requestAnimationFrame(Gameloop);
    update();
    render();
    stats.update();
}
Gameloop();
// Auxiliary functions
function rndInt(min, max) {
    return Math.floor(Math.random() * (max - min) + min) + 1;
}
function rndM(min, max) {
    return Math.random() * (max - min) + min;
}
```



2-drawing
A view of a spatial fractal with 1-2-3 iteration steps



3-drawing
A view of a spatial fractal with 1-2-3 iteration steps



4-drawing
A view of a spatial fractal with only the fourth iteration step

3. Research result

As a result of the research, a computer program has been developed for constructing a 3D model of the "Limit Cube of Cubes" fractal based on the iterative function system (IFS). The algorithm implements sequential scaling and spatial copying of cubic elements using cubic symmetry vectors.

The resulting model has the property of self-similarity and makes it possible to visualize the formation of a fractal structure at various iteration levels. It is established that with an increase in the number of iterations, the structure tends to the ultimate shape in the form of a cube, which acts as the limiting surface of the fractal.

Thus, the correctness of the mathematical model and the effectiveness of the applied algorithm for constructing and analyzing spatial fractal objects have been confirmed.

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Inverse Problem of Determining the Source of the Heat Equation with Fractional Time Derivatives

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Abstract

In this paper, we study a linear inverse problem for a heat conductivity equation with fractional time derivatives with overdetermination conditions at an interior point. First, we study the direct problem. Using the Fourier method, this direct problem is reduced to an equivalent integral equation. Then, using estimates of the Mittag-Leffler function and some of its properties, the inverse problem is reduced to an equivalent a linear system of integral equations. To solve this system, the Volterra integral equation method is used. Global theorems of existence and uniqueness of the solution of the inverse problem under consideration are proved.

Keywords: Fractional time equation, Fractional derivative, Fourier method, Mittag-Leffler function, Bessel inequality, Inverse problem, Integral equation, Volterra.
2020 MSC: 34A55, 34B05, 58C40.

1. Introduction

Inverse problems for differential equations are understood to mean problems of finding coefficients, right-hand sides, as well as initial or boundary conditions and solutions of differential equations given additional information (redetermination) about the solution of the direct problem.

Inverse problem theory is an important, independent field of research in the field of differential equations.

Problems for these equations in which the unknowns include the right-hand side of the differential equation, the coefficients of the differential operator, and the initial and boundary conditions are of great practical importance. These unknown elements of the initial and initial-boundary value problems are determined using additional (redefined) information about the solution of the direct (initial or initial-boundary value) problem.

The history of the development of the theory of inverse problems for second and third order equations, its current state, and a detailed bibliography are presented in the works [1-4, 9-11, 13, 15, 19, 22, 23, 26].

The theory of fractional partial differential equations has recently found significant application in various fields of science and engineering, particularly in physics, chemistry, the electrodynamics of complex media, polymer rheology, and other fields.

Inverse problems for fractional differential heat equations have been little studied. The most common problems encountered in the literature are those of determining a linear source and those of determining a coefficient for an initial-boundary value problem with various types of overdetermination conditions (see, for example, [38–43]).

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The diffusion equation with fractional time derivative was obtained by replacing the ordinary time derivative with a derivative with fractional time derivative. Direct problems, such as initial value problems and initial-boundary value problems for the diffusion equation and for pseudo-parabolic equations with fractional time derivatives, have been widely studied in recent years (see [4-8, 20, 21, 24, 28-30]).

Recently, the study of fractional differential equations has made significant progress. Several definitions of fractional integrals and derivatives exist in the literature, the most popular being those in the sense of Riemann–Liouville and Caputo derivatives. Direct problems for fractional partial differential equations are addressed in monographs [1-7].

Inverse problems for integer and fractional partial differential equations are a rapidly developing area of mathematics. While classical formulations of boundary value problems for these equations have already been sufficiently well studied and conditions for their solvability have been obtained, the situation with inverse problems is more complex. Often, even the very formulation of such problems requires additional research, including the study of the differential properties of solutions to direct problems.

2. Definition of Fractional Derivatives and Integrals

Let us introduce some concepts necessary for further research.

Definition 2.1. A fractional Caputo differential operator x of order Z for a differentiable function f is an operator defined by the expression [3,4]:

$$D_t^{\alpha}[f](t) = I\left[f'(t)\right] = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(\tau)(t-\tau)^{-\alpha} d\tau, & 0 < \alpha < 1, \\ f'(t), & \alpha = 1, \end{cases} \tag{2.1}$$

where $\Gamma(z)$ is the gamma function.

Definition 2.2. The fractional Riemann-Liouville integral operator $D_{0t}^{-\alpha}$ of order α, $0 < \alpha \leq 1$ for the integrable function f is the operator defined by the expression [3,4]:

$$D_{0t}^{-\alpha} f(t) = I^{\alpha}\left[f(t)\right] = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau, & 0 < \alpha < 1, \\ \int_0^t f(\tau) d\tau, & \alpha = 1. \end{cases} \tag{2.2}$$

Definition 2.3. A two-parameter function $E_{\alpha,\beta}(z)$ defined by the formula [3]:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha > 0, \quad \beta > 0) \tag{2.3}$$

is called the Mittag-Leffler function.

Let us present some relationships given in [3]-[6]:

$$E_{1,1}(z) = e^z, \quad E_{1,1}(z) = \frac{e^z - 1}{z}, \tag{2.4}$$

$$E_{2,1}(z) = ch\sqrt{z}, \quad E_{2,2}(z) = \frac{sh\sqrt{z}}{\sqrt{z}}, \tag{2.5}$$

$$E_{1/2,1}(z) = \frac{2}{\sqrt{\pi}} e^{-z} \mathrm{erfc}(-\sqrt{z}), \tag{2.6}$$

For $\beta = 1$ we obtain a single-parameter Mittag-Leffler function:

$$E_{\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \equiv E_\alpha(z). \tag{2.7}$$

Generalization of the Newton-Leibniz formula for α, $(0 < \alpha \leq 1)$.

$$D_{0t}^{-\alpha} D_t^\alpha z(t) = z(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} z^{(\alpha-1)}(0). \tag{2.8}$$

Lemma 2.1. ([9]) Let $0 < \alpha \leq 1$. Then there exist positive constants N_1, N_2, N_3:

$$\frac{N_1}{1+y} \leq E_{\alpha,1}(-y) \leq \frac{N_2}{1+y}, \quad E_{\alpha,\alpha}(-y) \leq \frac{N_3}{1+y}, \quad \text{for all } y \geq 0, \alpha \in \mathbb{R}. \tag{2.9}$$

Lemma 2.2. ([10]) If $0 < \alpha < 1$ and $\lambda > 0$, then

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0; \tag{2.10}$$

$$D_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0. \tag{2.11}$$

Lemma 2.3. ([1-4]) For $\lambda > 0$, $\alpha > 0$ and $m \in$, then

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0; \tag{2.12}$$

$$\frac{d}{dt}(t E_{\alpha,2}(-\lambda t^\alpha)) = E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0, \tag{2.13}$$

$$\frac{d}{dt}\left(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\right) = -t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha), \quad t > 0. \tag{2.14}$$

Lemma 2.4. ([1-4]). If $\alpha > 0$, $\beta > 0$ and $\lambda > 0$, then

$$\frac{d}{dt} t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha) = t^{\beta-2} E_{\alpha,\beta-1}(-\lambda t^\alpha), \quad t > 0. \tag{2.15}$$

The proof of these lemmas follows from the definition of the Caputo fractional derivative and the differentiation of the two-parameter Mittag-Leffler function.

Lemma 2.5. (see [12]). If for $0 < \alpha < 1$ and $\lambda > 0$ the function $g(t) \in AC[0,T]$, then

$$\begin{aligned} D_t^\alpha \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k^2(t-\tau)^\alpha) g(\tau) d\tau = g(t) - \\ -\lambda \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k^2(t-\tau)^\alpha) g(\tau) d\tau, \quad 0 < t \leq T. \end{aligned} \tag{2.16}$$

In particular, if $\lambda = 0$, then

$$D_t^\alpha \int_0^t (t-\tau)^{\alpha-1} g(\tau) d\tau = g(t)\Gamma(\alpha), \quad 0 < t \leq T. \tag{2.17}$$

Let $\Omega_T = \{(x,t) : 0 < x < \pi, 0 < t \leq T\}$, $T > 0$ be a fixed number.

3. Statement of the Problem and it's Study

Consider in domain Ω_T the inverse problem of finding the temperature distribution $u(x,t)$ and the source function $f(t)$ from the following system:

$$D_t^\alpha u(x,t) = u_{xx}(x,t) + f(t)h(x,t) + g(x,t), \quad (x,t) \in \Omega_T, \tag{3.1}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{3.2}$$

$$u(0,t) = u(\pi,t) = 0, \quad t \in [0,T], \tag{3.3}$$

$$u(x_0,t) = \psi(t), \quad 0 \le t \le T, \quad 0 < x_0 < l, \tag{3.4}$$

where $\varphi(x)$ and $h(x,t)$ are given functions.

Here D_t^α is the Caputo fractional derivative of order α $(0 < \alpha \le 1)$.

Definition 3.1. A function $u(x,t)$ is called a classical solution of the initial-boundary value problem (3.1)-(3.3), if

1) $u(x,t)$ is twice continuously differentiable with respect to x for all $t \in [0,T]$,

2) for each $x \in (0,l)$ the function $u(x,t)$ is continuously differentiable with respect to t in $[0,T]$ and its fractional integral

$$\frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau$$

is continuously differentiable with respect to t in $[0,T]$

3) function $u(x,t)$ satisfies conditions (3.1)-(3.3).

Definition 3.2. A pair of functions $u(x,t)$ and $f(t)$ is called a solution to the inverse problem (3.1)-(3.4) if $u(x,t) \in C^{(2,1)}(\Omega) \cap C(\bar{\Omega})$, $f(t) \in C[0,T]$ and satisfy equalities (3.1)-(3.4) in the classical sense.

Condition (3.4) is called overdetermination at an interior point.

Inverse problems with this type of overdetermination condition for the heat equation and the third-order pseudo parabolic equation were studied in [6-8].

Theorem 3.1. *Let* $\varphi(x) \in C^2[0,\pi]$, $\varphi''(x) \in L_1(0,\pi)$, $\varphi(0) = \varphi(\pi) = 0$, $\varphi''(0) = \varphi'(\pi) = 0$, $h(x,t) \in C^{(2,0)}(\Omega_T)$, $h_{xxx}(x,t) \in L_2(\Omega_T)$, $|h(x_0,t)| \ge \alpha > 0$, $h(0,t) = h(0,\pi) = 0$, $h''_{xx}(0,t) = h''_{xx}(\pi,t) = 0$, $g(x,t) \in C^{(2,0)}(\Omega_T)$, $g_{xxx}(x,t) \in L_2(\Omega_T)$, $g(0,t) = g(0,\pi) = 0$, $g''_{xx}(0,t) = g''_{xx}(\pi,t) = 0$ *and, in addition, the compatibility conditions* $\varphi(x_0) = \psi(0)$ *are satisfied. Then for any* $T > 0$ *the solution of the inverse problem (3.1) - (3.4) on the interval* $[0,T]$ *exists, is unique, and belongs to the class* $C[0,T]$.

Proof. Note that since problem (3.1) - (3.4) is linear, its solution can be sought in the form

$$\{u(x,t), f(t)\} = \{v(x,t), 0\} + \{w(x,t), f(t)\}, \tag{3.5}$$

where $v(x,t)$ is the solution in Ω_T of the direct problem

$$\begin{cases} D_t^\alpha v(x,t) = v_{xx}(x,t) + g(x,t), \quad (x,t) \in \Omega_T, \\ v(x,0) = \varphi(x), \quad 0 \le x \le \pi, \\ v(0,t) = v(\pi,t) = 0, \quad t \in [0,T]. \end{cases} \tag{3.6}$$

and the pair $\{w(x,t), f(t)\}$ is the solution in Ω_T of the inverse problem

$$\begin{cases} D_t^\alpha w(x,t) = w_{xx}(x,t) + f(t)h(x,t), \quad (x,t) \in \Omega_T, \\ w(x,0) = 0, \quad 0 \le x \le \pi, \\ w(0,t) = w(\pi,t) = 0, \quad t \in [0,T], \\ w(x_0,t) = \psi(t) - v(x_0,t), \quad 0 \le t \le T, \quad 0 < x_0 < \pi. \end{cases} \tag{3.7}$$

It is known that the solution to problem (3.6) has the form (see [1])

$$v(x,t) = \sum_{n=1}^{\infty} \sin nx \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}(-n^2(t-\tau)^\alpha) g_n(\tau) d\tau + \sum_{n=1}^{\infty} E_{\alpha,1}(-n^2 t^\alpha) \varphi_n \sin nx, \tag{3.8}$$

where

$$\varphi_n = \frac{2}{\pi} \int_0^\pi \varphi(\xi) \sin n\xi d\xi, \quad g_n(t) = \frac{2}{\pi} \int_0^\pi g(\xi,t) \sin n\xi d\xi. \tag{3.9}$$

In formula (3.9), integrating by parts three times, we obtain

$$\varphi_n = \frac{2}{\pi} \int_0^\pi \varphi(\xi) \sin n\xi d\xi = \frac{2}{\pi} \frac{1}{n^3} \int_0^\pi \varphi'''(\xi) \cos n\xi d\xi,$$

$$h_n(t) = \frac{2}{\pi} \int_0^\pi h(\xi,t) \sin k\xi d\xi = \frac{2}{\pi} \frac{1}{n^3} \int_0^\pi h'''_{xxx}(\xi,t) \cos n\xi d\xi.$$

Hence, by virtue of Bessel's inequality, the series of squares $|\varphi''(\xi)|^2$ and $|g'''_{xxx}(\xi,t)|^2$ converge, therefore $|\varphi_n| \to 0$ and $|g_n| \to 0$ when $n \to \infty$. Consequently, the series (3.8) converges uniformly and absolutely, when $t \ge 0$.

Therefore, from the conditions of Theorem 3.1 it follows that the solution of the direct problem (3.6) exists and is unique. Therefore, to prove Theorem 3.1 it is sufficient to prove the existence and uniqueness of the definition of a pair of functions $\{w(x,t), f(t)\}$ from conditions (3.7). Therefore, without loss of generality, in problem (3.1) - (3.4), we assume that $\varphi(x) = g(x,t) = 0$.

Thus, it is enough for us to prove the existence and uniqueness of the solution to the inverse problem of determining a pair of functions $\{u(x,t), f(t)\}$ from the conditions

$$D_t^\alpha u(x,t) - u_{xx}(x,t) = f(t)h(x,t), \quad 0 < x < \pi, \quad 0 < t \le T, \tag{3.10}$$

$$u(x,0) = 0, \quad 0 \le x \le \pi, \tag{3.11}$$

$$u(0,t) = u(\pi,t) = 0, \quad 0 \le t \le T, \tag{3.12}$$

$$u(x_0,t) = \psi(t), \quad 0 \le t \le T, \quad 0 < x_0 < \pi. \tag{3.13}$$

Let us now introduce into consideration the functions $z(x,t) = u_{xx}(x,t)$. Then the pair $\{z(x,t), f(t)\}$ will satisfy the conditions

$$D_t^\alpha z(x,t) - z_{xx}(x,t) = f(t)h_{xx}(x,t), \quad 0 < x < \pi, \quad 0 < t \le T, \tag{3.14}$$

$$z(x,0) = 0, \quad 0 \le x \le \pi, \tag{3.15}$$

$$z(0,t) = z(\pi,t) = 0, \tag{3.16}$$

$$f(t) = \left[D_t^\alpha \psi(t) - z(x_0,t)\right] / h(x_0,t), \quad 0 < t \le T, \tag{3.17}$$

We find the solution of the inhomogeneous equation (3.14) in the form of a Fourier series expansion in terms of the eigenfunctions $\sin(nx)$ of the corresponding eigenvalue problem:

$$z(x,t) = \sum_{n=1}^{\infty} z_n(t)\sin(nx), \tag{3.18}$$

where

$$z_n(t) = \frac{2}{\pi}\int_0^{\pi} z(x,t)\sin nxdx.$$

Let us expand the function $h_{xx}(x,t)$ into a Fourier series

$$h_{xx}(x,t) = \sum_{n=1}^{\infty} h_n(t)\sin(nx), \tag{3.19}$$

where

$$h_n(t) = \frac{2}{\pi}\int_0^{\pi} h_{xx}(x,t)\sin nxdx.$$

Substituting (3.18), (3.19) into equation (3.14), we obtain

$$\sum_{n=1}^{\infty}\left\{D_t^{\alpha} z_n(t) + n^2 z_n(t) - h_n(t)f(t)\right\}\sin(nx) = 0. \tag{3.20}$$

From here, due to the completeness of the $\{\sin(nx)\}$ system and from (3.20), to find the function $z_n(t)$ we obtain the following Cauchy problem:

$$D_t^{\alpha} z_n(t) + n^2 z_n(t) = f(t)h_n(t), \tag{3.21}$$

$$z_n(0) = 0. \tag{3.22}$$

Problem (3.21), (3.22) has a unique solution and this solution can be represented in the form

$$z_n(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-n^2(t-\tau)^{\alpha})h_n(\tau)f(\tau)d\tau, \quad n = 1,2,\ldots. \tag{3.23}$$

For further research, we prove the following lemma. □

Lemma 3.1. For sufficiently large values n of n and $t \in [0,T]$, the following estimates hold:

$$|z_n(t)| \le \frac{t^{\alpha}\|h_n\|\|f\|}{\Gamma(\alpha+1)}, \quad n = 1,2,\ldots \tag{3.24}$$

$$|D_t^{\alpha} z_n(t)| \le \frac{\|h_n\|\left(t^{\alpha}\|f\| + \Gamma(\alpha+1)\right)}{n^2}, \quad n = 1,2,\ldots \tag{3.25}$$

Proof. Directly estimating the function $z_n(t)$ for $0 \le t \le T$ and taking into account the inequality $0 \le E_{\alpha,\alpha}(-\sigma) \le \Gamma(\alpha)$ (see [14], pp. 44-45), we obtain

$$|z_n(t)| \le \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-a_n(t-\tau))|f(\tau)||h_n(\tau)|d\tau \le$$

$$\leq \frac{t^\alpha}{\Gamma(\alpha+1)} \max_{0\leq t\leq T} |f(t)| \max_{0\leq t\leq T} |h_n(t)| \leq \frac{t^\alpha \|h_n\|\|f\|}{\Gamma(\alpha+1)}, \quad n = 1, 2, \ldots$$

where

$$\|h_n(t)f(t)\| = \max_{0\leq t\leq T} |h_n(t)f(t)| = \|h_n\|\|f\|.$$

From (3.21), taking into account the estimate (3.24), we have

$$|D_t^\alpha z_n(t)| \leq |-z_n(t)| + |h_n(t)| \leq \|h_n\|\|f\| \left(\frac{t^\alpha + \Gamma(\alpha+1)}{\Gamma(\alpha+1)}\right).$$

Substituting the expressions for $z_n(t)$ $n = 1, 2, \ldots$ into (3.18), we have

$$z(x,t) = \sum_{n=1}^{\infty} \sin nx \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}(-n^2(t-\tau)^\alpha) h_n(\tau) f(\tau) d\tau. \tag{3.26}$$

Therefore, the function $z(x,t)$ defined by series (3.26) is continuous in domain $\bar{\Omega}_T$ and satisfies the initial and boundary conditions. It is easy to show that the function $z(x,t)$ satisfies equation (3.14) in domain Ω_T. To do this, it suffices to demonstrate the uniform convergence of the series $\sum_{n=1}^{\infty} D_t^\alpha z_n$, $\sum_{n=1}^{\infty} \frac{\partial^2 z_n}{\partial x^2}$.

From the conditions imposed on the functions $h_{xx}(x,t)$ it follows that

$$|h_n(t)| \leq \frac{\text{const}}{n^4}.$$

Hence it follows that the series $\sum_{n=1}^{\infty} z_n$, $\sum_{n=1}^{\infty} D_t^\alpha z_n$, $\sum_{n=1}^{\infty} \frac{\partial^2 z_n}{\partial x^2}$ converge uniformly and absolutely.

Next, we prove the existence and uniqueness of a solution to the inverse problem (3.1) - (3.4). Let $f(t) \in C[0,T]$ be a solution to the inverse problem (3.1) - (3.4). Substituting (3.8) and (3.26) into (3.5), we obtain

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \sin nx \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}(-n^2(t-\tau)^\alpha) g_n(\tau) d\tau + \sum_{n=1}^{\infty} E_{\alpha,1} \times \\ \times(-n^2 t^\alpha)\varphi_n \sin nx + \sum_{n=1}^{\infty} \sin nx \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}(-n^2(t-\tau)^\alpha) h_n(\tau) f(\tau) d\tau. \end{aligned} \tag{3.27}$$

Differentiating relation (3.27) by the operator D_t^α and taking into account formulas (3.26), and using the overdetermination (3.4), we find

$$D_t^\alpha z\Big|_{x=x_0} = h(x_0,t)f(t) - \sum_{n=1}^{\infty} n^2 \sin nx_0 \int_0^t (t-\tau)^\alpha E_{\alpha,1}(-n^2(t-\tau)^\alpha) h_n(\tau) f(\tau) d\tau = D_t^\alpha \psi(t).$$

or

$$f(t) - \int_0^t K(t-\tau) f(\tau) d\tau = \psi_1(t), \tag{3.28}$$

where

$$\psi_1(t) = \left[D_t^{\alpha}\psi(t) + \sum_{n=1}^{\infty} n^2 E_{\alpha,1}(-n^2 t^{\alpha})\varphi_n \sin nx_0 \right] / h(x_0, t)$$

$$K(t) = \frac{1}{h(x_0,t)} \sum_{k=1}^{\infty} k^2 t^{\alpha} E_{\alpha,1}(-k^2 t^{\alpha}) h_k(t) \sin kx_0.$$

From the conditions imposed on the functions $f_h(x,t)$ it follows that the functions $\psi_1(t), K(t)$ are continuous.

Therefore, integral equation (3.27) is a Volterra integral equation of the second kind. Consequently, equation (3.27) has a unique solution $f(t) \in C[0,T]$.

Next, substituting the function $f(t)$ into (3.10), we uniquely find the function $u(x,t)$. As in [8], this function is a solution to the inverse problem (3.1) - (3.4). Theorem 3.1 is proven. □

4. Conclusion

In this paper, we investigated the unique solvability of a solution to a time-dependent inverse source determination problem, including a heat equation with a fractional time derivative, boundary conditions, and overdetermination conditions at an interior point.

Using the Fourier method and Volterra integral equations, we proved the existence and uniqueness theorem for the solution to this inverse problem.

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Constructing Solutions to a Fourth-Order Partial Integro-Differential Equation

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Abstract

There is considered solution of an integro-differential equation in which the coefficients of the derivatives of the unknown function up to the fourth order inclusive are known functions. This equation is studied in an operator form, and the solution to the original problem is obtained using the method of additional argument. The prominent feature of this methodology is its ability to determine analytical solutions in some cases. Overall, the method of additional argument represents a robust contemporary framework capable of producing mathematically rigorous results.

Keywords: differential, integral, reduction, additional argument, operator, notation, partial derivatives.
2020 MSC: 45K05, 35G15, 35C05.

1. Introduction

There are many works devoted to studying solutions of first- and second-order partial differential equations. Among them, the studies by authors [21-25] stand out. At the same time, the construction of solutions for third- and higher-order equations has been investigated to a much lesser extent. Nevertheless, in recent years it has become possible to identify a number of works [1, 2, 4-12, 17-20] focused on the study of solutions to such partial differential equations. The complexity of finding solutions escalates significantly with the increasing order of the equation.

Currently, new methods are being developed to reduce the initial value problem for higher-order differential equations to integral equations. One such approach is the so-called method of an additional argument (MAA). Extensions of this method to new classes of equations were made in [3, 13, 14, 15].

The MAA method can be used to search solutions of various types of equations and conditions. Nevertheless, much of the current research focuses specifically on solving equations with given initial conditions. One of the key challenges is applying this method to boundary value problems.

Developing solutions for higher-order partial differential equations using established methods presents a more complex challenge. The problem of constructing a solution using MAA is discussed in [16]. A solution to the problem under consideration has developed, and it has proven that this solution is unique and meets the specified requirements.

This study investigates the problem of constructing solutions to integro-differential equations using the MAA, where the coefficients of the derivatives of the unknown function up to

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the fourth order are given functions. It is also worth noting that the use of MAA reduces the problem to an algebraic equation. The obtained results are validated by solving a specific example.

2. Material and Methods

Formulation of the problem. The Cauchy problem for the smooth function $u(t,x)$ is considered:

$$D^2[-a(t,x)]D^2[a(t,x)]u(t,x) = f(t,x) + H(t)W[u],$$

$$W[u] := \int_0^1 \int_0^1 K(s,\xi)u(s,\xi)dsd\xi, \quad (t,x) \in G := R_+ \times R, \tag{2.1}$$

$$u(0,x) = \psi_0(x), x \in R, \tag{2.2}$$

$$\frac{\partial^k u(0,x)}{\partial t^k} = \lambda_k(x), \quad k = 1,2,3, x \in R, \tag{2.3}$$

$$a(t,x), f(t,x) \in \overline{C}^{(4)}(G), \quad \psi_0(x), \quad \lambda_k(x) \in \overline{C}^{(4)}(R), \quad (k=1,2,3), \quad H(t) \in \overline{C}^{(4)}(R_+),$$

$$K(s,\xi) \in C([0,1] \times [0,1]), \quad D[\omega] = \frac{\partial}{\partial t} + \omega\frac{\partial}{\partial x},$$

$\overline{C}^{(k)}(\Omega)$ is the space of real-valued functions being continuous and bounded with the derivatives up to the kth order.

3. The Research Findings and Discussion

Rewrite the equation (2.1) in the form:

$$D^2[-a(t,x)]D^2[a(t,x)]u(t,x) = f(t,x) + W[u] \cdot H(t). \tag{3.1}$$

For convenience, we introduce the following new functions based on conditions (2.2) and (2.3):

$$D[-a(t,x)]D^2[a(t,x)]u(t,x)\big|_{t=0} = \varphi_1(x),$$

$$D^2[a(t,x)]u(t,x)\big|_{t=0} \equiv \varphi_0(x),$$

$$D[a(t,x)]u(t,x)\big|_{t=0} = \psi_1(x),$$

These functions $p(s,t,x)$, $q(s,t,x)$ can be obtained by solving the following integral equations.

$$p(s,t,x) = x + \int_s^t a(\nu, p(\nu,t,x))d\nu, \tag{3.2}$$

$$q(s,t,x) = x - \int_s^t a(\nu, q(\nu,t,x))d\nu, \tag{3.3}$$

$$(s,t,x) \in Q := \{0 \le s \le t < \infty, x \in R\}.$$

It is essential to mention that integral equations (3.2) and (3.3) possess a unique solution based on the properties of the function under the following conditions $a(t,x) \in \overline{C}^{(4)}(G)$:

$$p(s,s,x) = x, \quad q(s,s,x) = x.$$

The integral equations (3.2) and (3.3) give the following equations:

$$D[-a(t,x)]p(s,t,x) = 0, \tag{3.4}$$

$$D[a(t,x)]q(s,t,x) = 0. \tag{3.5}$$

Lemma 3.1. The Cauchy problem (3.1), (2.2), and (2.3) can be reduced to the following integral equation, and their solutions are equivalent:

$$u(t,x) = \sum_{k=0}^{1} \psi_k(q(0,t,x))\frac{t^k}{k!} + \int_0^t (t-s)\sum_{k=0}^{1} \varphi_k(p(0,s,q(s,t,x))\frac{s^k}{k!}ds + \int_0^t (t-s)\times$$

$$\times \int_0^s (s-\tau)f(\tau, p(\tau,s,q(s,t,x)))d\tau ds + W[u]\int_0^t (t-s)\int_0^s (s-\tau)H(\tau)d\tau ds.$$

Proof. We can note: $z(t,x;u) = D^2[a(t,x)]u(t,x), \quad b(t,x) = -a(t,x)$, then (3.1) takes the following form:

$$D^2[b(t,x)]z(t,x;u) = f(t,x) + W[u]\cdot H(t). \tag{3.6}$$

if we introduce the following notation $z_1(t,x;u) = D[b(t,x)]z(t,x;u)$, then the equation (3.1) can be written as follows:

$$D[b(t,x)]z_1(t,x;u) = f(t,x) + W[u]\cdot H(t). \tag{3.7}$$

Let's use the MAA to the problem (3.7), (2.2), (2.3). As a result, we obtain the following integro-differential equation:

$$z_1(t,x;u) = \varphi_1(p(0,t,x)) + \int_0^t f(s,p(s,t,x))ds + W[u]\int_0^t H(s)ds. \tag{3.8}$$

In fact, by differentiating (3.8), we get:

$$D[b(t,x)]z_1(t,x;u) = \varphi_1'(p(0,t,x))D[b(t,x)]p(0,t,x) + f(t,x) + W[u]\cdot H(t). \tag{3.9}$$

Using equation (3.4) from (3.9), we obtain equation (3.7).
If we put $t = 0$ in equation (3.8), then $z_1(0,x;u) = \varphi_1(x)$.
Now we use the MAA for equation (3.8).

$$z(t,x;u)=\sum_{k=0}^{1}\varphi_k(p(0,t,x))\frac{t^k}{k!}+\int_0^t(t-s)f(s,p(s,t,x))ds+W[u]\int_0^t(t-s)H(s)ds. \quad (3.10)$$

Equation (3.10) satisfies (3.8) and initial conditions (2.2), (2.3).

In fact, by differentiating (3.10), we get:

$$D[b(t,x)]z(t,x;u)=z_1(t,x;u).$$

So we get (3.10) by introducing a new function $z_1(t,x;u)$. If on the contrary we use the operator $D[b(t,x)]$ for the equality of (3.10) twice in a row, it can be seen (3.1), (2.2), (2.3),

Let's introduce some more notation:

$$\theta(t,x;u)=D[a(t,x)]u(t,x),$$

Then the equation (3.10) becomes as follows:

$$D[a(t,x)]\theta(t,x;u)=$$

$$=\sum_{k=0}^{1}\varphi_k(p(0,t,x))\frac{t^k}{k!}+\int_0^t(t-s)f(s,p(s,t,x))ds+W[u]\int_0^t(t-s)H(s)ds. \quad (3.11)$$

Let's apply the MAA to the problem (3.11), (2.2), (2.3). We obtain the following equation:

$$\theta(t,x;u)=\psi_1(q(0,t,x))+\int_0^t\sum_{k=0}^{1}\varphi_k(p(0,s,q(s,t,x)))\frac{s^k}{k!}ds+$$

$$+\int_0^t\int_0^s(s-\tau)f(\tau,p(\tau,s,q(s,t,x)))d\tau ds+W[u]\int_0^t\int_0^s(s-\tau)H(\tau)d\tau ds. \quad (3.12)$$

In fact, we can prove (3.12) by differentiation:

$$D[a(t,x)]\theta(t,x;u)=\psi_1'(q)D[a(t,x)]q(0,t,x)+$$

$$+\int_0^t\sum_{k=0}^{1}\varphi_k'(p(0,s,q))\frac{\partial p}{\partial x}D[a(t,x)]q(s,t,x)ds+$$

$$+\int_0^t\int_0^s(s-\tau)\frac{\partial f}{\partial x}\frac{\partial p}{\partial x}D[a(t,x)]q(s,t,x)d\tau ds+\sum_{k=0}^{n-1}\varphi_k(p(0,t,x))\frac{t^k}{k!}+$$

$$+\int_0^t(t-s)f(s,p(s,t,x))ds+W[u]\int_0^t(t-s)H(s)ds.$$

Considering (3.5) from the last equation, there is followed equation (3.11). Also, if we put $t=0$ in equation (3.11), then $\theta(0,x;u)=\psi_1(x)$.

For problems (3.12), (2.2), (2.3), we will once again use the MAA:

$$u(t,x)=\sum_{k=0}^{1}\psi_k(q(0,t,x))\frac{t^k}{k!}+\int_0^t(t-s)\sum_{k=0}^{1}\varphi_k(p(0,s,q(s,t,x))\frac{s^k}{k!}ds+\int_0^t(t-s)\times$$

$$\times \int_0^s (s-\tau) f(\tau, p(\tau, s, q(s,t,x))d\tau ds + W[u] \int_0^t (t-s) \int_0^s (s-\tau) H(\tau) d\tau ds. \qquad (3.13)$$

In fact, by differentiating (3.13), we get:

$$D[a(t,x)]u(t,x) = [\psi_0'(q) + \psi_1'(q)t] D[a(t,x)]q(0,t,x) +$$

$$+ \int_0^t (t-s) \sum_{k=0}^{1} \varphi_k'(p(0,s,q)) \frac{\partial p}{\partial x} D[a(t,x)]q(s,t,x) +$$

$$+ \int_0^t (t-s) \int_0^s (s-\tau) \frac{\partial f}{\partial x} \frac{\partial p}{\partial x} D[a(t,x)]q(s,t,x) d\tau ds + \theta(t,x;u).$$

From the last, we get (3.12) with using the equality (3.5).
The initial condition is satisfied in (3.13): $t = 0,\ u(0,x) = \psi_0(x)$. □

Lemma 3.2. Let's

$$W_1 := \int_0^\infty \int_{-\infty}^\infty K(s,\xi) \int_0^s (s-\eta) \int_0^\eta (\eta - \tau) H(\tau) d\tau d\eta ds d\xi \neq 1.$$

The only solution to problem (2.1), (2.2), (2.3) exists in Q and is defined as follows:

$$u(t,x) = \sum_{k=0}^{1} \psi_k(q(0,t,x)) \frac{t^k}{k!} + \int_0^t (t-s) \sum_{k=0}^{1} \varphi_k(p(0,s,q(s,t,x)) \frac{s^k}{k!} ds + \int_0^t (t-s) \times$$

$$\times \int_0^s (s-\tau) f(\tau, p(\tau, s, q(s,t,x))d\tau ds + \frac{B}{1-W_1} \int_0^t (t-s) \int_0^s (s-\tau) H(\tau) d\tau ds, \qquad (3.14)$$

here

$$B = \int_0^\infty \int_0^\infty K(s,\xi) \Big\{ \sum_{k=0}^{1} \psi_k(q(0,s,\xi)) \frac{s^k}{k!} + \int_0^s (s-\tau) \sum_{k=0}^{1} \varphi_k(p(0,\tau,q(\tau,s,\xi)) \frac{\tau^k}{k!} d\tau +$$

$$+ \int_0^s (s-\tau) \int_0^\eta (\eta - \tau) f(\tau, p(\tau, \eta, q(\eta, s, \xi))) d\tau d\eta \Big\} ds d\xi.$$

Proof. It is necessary to determine the value of $W[u]$ in order for (3.13) to constitute a complete solution of the equation. For that, we multiply both sides of equation by the function $K(t,x)$ and integrate from 0 to 1 with respect to t, and from 0 to 1 with respect to x.

As a result, we get the following algebraic equation:

$$W[u] = B + W[u]W_1.$$

From this we obtain:

$$W[u] = \frac{B}{1-W_1}, \quad W_1 \neq 1.$$

Let's put (3.14) into (3.13). As a result, we get (3.13). Lemma 3.2 is proved.
Let us consider an example. □

Example. Let's $a(t,x)=1$, $H(t)=1$, $K(s,\xi)=1$ in the equation (2.1):

$$u_{tttt}(t,x)-2u_{ttxx}(t,x)+u_{xxxx}(t,x)=t+\int_0^1\int_0^1 u(s,\xi)\,d\xi\,ds. \tag{3.15}$$

Consider equation (3.15) with the following conditions:

$$u(0,x)=x^2,$$
$$u_t(0,x)=0,$$
$$u_{tt}(0,x)=x,$$
$$u_{ttt}(0,x)=0.$$

For convenience, we introduce new notation:

$$\overline{D}=\frac{\partial}{\partial t}-\frac{\partial}{\partial x},\quad D=\frac{\partial}{\partial t}+\frac{\partial}{\partial x},$$
$$\overline{D}^2D^2u(t,x)=t+\int_0^1\int_0^1 u(s,\xi)\,d\xi\,ds,$$
$$\overline{D}D^2u(t,x)\big|_{t=0}=[u_{ttt}+u_{ttx}-u_{txx}-u_{xxx}]\big|_{t=0}=1=\varphi_1(x),$$
$$D^2u(t,x)\big|_{t=0}=[u_{tt}+2u_{tx}+u_{xx}]\big|_{t=0}=x+2=\varphi_0(x),$$
$$Du(t,x)\big|_{t=0}=[u_t+u_x]\big|_{t=0}=2x=\psi_1(x),$$
$$u(0,x)=x^2=\psi_0(x).$$

As a result, we get the following solution:

$$u(t,x)=x^2+x\frac{t^2}{2}+\frac{t^5}{5!}+\frac{43}{102}\cdot\frac{t^4}{4!}.$$

4. Conclusion

The solution (3.14) of the Cauchy problem (2.1), (2.2), (2.3) for the fourth-order partial integro-differential equation, in which the coefficients of the derivatives of the unknown function up to the fourth order are known functions, has derived using the method of an additional argument.

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Implementation of a Regularization-Based Numerical Method for Solving the First-Kind Volterra Integral Equation Using Python and Numpy

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Abstract

In this paper, the problem of solving the Volterra integral equation of the first kind, which belongs to the class of ill-posed problems, is considered. Based on the method of regularization and quadrature approximation, a stable numerical algorithm was created. A modern software implementation using the Python programming language and the NumPy library is presented, which provides efficient numerical processing and computational stability of the algorithm. The stability, computational complexity, and accuracy of the algorithm have been analyzed. The results of numerical experiments confirm the effectiveness of the proposed method in solving inverse problems of mathematical physics.

Keywords: Volterra equation of the first kind, regularization, Python, NumPy, numerical methods, ill-posed problem, integral equations, inverse problems, computational experiment.
2020 MSC: 45D05, 65R20, 65J20.

1. Introduction

Volterra integral equations of the first kind are widely used in modeling of heat conduction processes, geophysical exploration, signal processing, and solving of inverse problems. A distinctive feature of these equations is their Hadamard's ill-posedness, as a result of which small errors in the initial data can lead to significant deviations in the solution. To ensure the stability of the computational process, regularization methods are applied, which allow transforming the original problem into a stable form. Despite a significant body of research dedicated to first-kind integral equations, the problem of developing efficient computational algorithms remains highly relevant [1, 2, 6].

2. Regularization-numerical algorithm

The purpose of this work is to develop a regularization-numerical algorithm for solving the Volterra integral equation of the first kind and to investigate its stability based on the difference approximations.

Furthermore, a numerical method to solve this equation was developed based on the constructed systematic algorithms, with the resulting difference (grid) schemes being stable

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[4]. Let us consider the Volterra integral equation of the first kind:

$$\int_0^x K(x,t)z(t)dt = f(x), \tag{2.1}$$

where $K(x,t)$ – is the known kernel of the integral operator, $f(x)$ is a given function, and $(z(x))$ represents the solution to be found. Since the problem is ill-posed, a regularization parameter $\varepsilon > 0$ is introduced to construct a stable computational algorithm.
where $K\,|_{t=x} = 0, \quad K_t(t,t) \geq \alpha > 0\,; \quad f(0) = 0$. Consequently, we have:

$$\begin{cases} \int\limits_0^x K_t(t,t)\psi(t)dt + \int\limits_0^x K_0(x,t)\psi(t)dt = f(x), \\ \int\limits_0^x z(t)dt = \psi(x), \quad \psi(0) = 0, \end{cases} \tag{2.2}$$

for simplicity: $z(0) = 0, K_0 \equiv K_t(x,t) - K_t(t,t)$, (if $z(0) = q$, then taking into account $z = V + q$, we will receive $V(0) = 0$. Therefore, without loss of generality, we require $z(0) = 0$).

In the papers [7-8] a systemic algorithm for regularization of the system was developed, and regularizability of (2.2) in a space with a uniform metric was proven, i.e.:
$(\psi_\varepsilon, z_\delta) \xrightarrow[\varepsilon\to 0(\delta\to 0)]{} (\psi, z), \quad (\varepsilon, \delta-$ are small parameters).

Therefore, based on system (2.2) and taking into account the parameters ε, δ, the following numerical solution method is provided in this paper:

$$\begin{cases} \varepsilon\psi_i + \sum\limits_{j=1}^{i} K_{0j}\psi_j h + \sum\limits_{j=1}^{i} N_{ij}\psi_j h = -f_i, \\ \delta z_i + \sum\limits_{j=1}^{i} z_j h = \psi_{\varepsilon i}, \quad i = \overline{1,n}. \end{cases}$$

$$\varepsilon\psi_{i-1} + \sum_{j=1}^{i} K_{0j-1}\psi_{j-1}h + \sum_{j=1}^{i} N_{ij-1}\psi_{j-1}h = \tilde{f}(s_i) \quad (i = \overline{1,n}) \tag{2.3}$$

and denote as $\tilde{\psi}^{\varepsilon,h} = \left\{\tilde{\psi}_{i-1}^{\varepsilon,h}\right\}, \quad (i = \overline{1,n})$ the solution of (2.3), and we introduce the vector $\tilde{\beta}^{\varepsilon,h} = \left\{\tilde{\beta}_{i-1}^{\varepsilon,h}\right\} \equiv \left\{\psi(s_{i-1}) - \tilde{\psi}_{i-1}^{\varepsilon,h}\right\}, \quad i = \overline{1,n.}$ We need the estimation of $\left\|\tilde{\beta}^{\varepsilon,h}\right\|_{C_n}$. For this we have:

$$\varepsilon\psi(s_{i-1}) + \sum_{j=1}^{i} K_{0j-1}\psi(s_{j-1})h + \sum_{j=1}^{i} N_{i\,j-1}\psi(s_{j-1})h = -f_i - \tilde{f}(s_i) - R_i + \varepsilon\psi(s_{i-1}) \tag{2.4}$$

where

$$R_i = \int_0^{x_i} N(x_i,s)\psi(s)ds - h\sum_{j=1}^{i} N_{i,j-1}\psi(s_{j-1}) \tag{2.5}$$

is the remainder of the quadrature of the right rectangles. Therefore, subtracting (2.3) from (2.4), we have:

$$[\varepsilon + hK_{0i\,j-1} + hN_{i,j-1}]\tilde{\beta}^{\varepsilon,h}_{i-1} + h\sum_{j=1}^{i-1} K_{0i\,j-1}\tilde{\beta}^{\varepsilon,h}_{i-1} + h\sum_{j=1}^{i-1} N_{i,j-1}\tilde{\beta}^{\varepsilon,h}_{j-1} =$$

$$= f(s_i) - \tilde{f}(s_i) - R_i + \varepsilon\psi(s_{i-1}). \tag{2.6}$$

Finding the difference between adjacent rows of the triangular system of linear algebraic equations (SLAE) (2.6) and proceeding to the estimate, we obtain:

$$\left|\tilde{\beta}^{\varepsilon,h}_{i-1}\right| \le \frac{\varepsilon}{\varepsilon+hK_1+hK_2}\left|\tilde{\beta}^{\varepsilon,h}_{i-2}\right| + \frac{h^2L_1}{\varepsilon+hK_1+hK_2}\sum_{j=1}^{i-1}\left|\tilde{\beta}^{\varepsilon,h}_{j-1}\right| + \frac{2\tilde{\delta}+|R_i-R_{i-1}|+\varepsilon|\psi(s_{i-1})-\varphi(s_{i-2})|}{\varepsilon+hK_1+hK_2} + \\ + \frac{h^2L_2}{\varepsilon+hK_1+hK_2}\sum_{j=1}^{i-1}\left|\tilde{\beta}^{\varepsilon,h}_{j-1}\right|, \tag{2.7}$$

$L_1 = \max\limits_{[0,b]\times R}|K'_x(x,s)|$, $L_2 = \max\limits_{[0,b]\times R'}|N'_x(x,s)|$, $K_1 = \min\limits_{[0,b]\times R}|K(s,s)|$, $K_2 = \min\limits_{[0,b]\times R}|N(s,s)|$, because $|R_i - R_{i-1}| \le ch^2$, $c = const$ [1].
Since $|\psi(s_{i-1}) - \psi(s_{i-2})| \le \Phi_1 h$, $\Phi_1 = \max|\psi'(s)|$, from (7) the inequality follows:

$$\left|\tilde{\beta}^{\varepsilon,h}_{i-1}\right| \le \frac{\varepsilon}{\varepsilon+hK_1+hK_2}\left|\tilde{\beta}^{\varepsilon,h}_{i-2}\right| + \frac{h^2L_1}{\varepsilon+hK_1+hK_2}\sum_{j=1}^{i-1}\left|\tilde{\beta}^{\varepsilon,h}_{j-1}\right| + \frac{h^2L_2}{\varepsilon+hK_1+hK_2}\sum_{j=1}^{i-1}\left|\tilde{\beta}^{\varepsilon,h}_{j-1}\right| + \\ + \frac{2\tilde{\delta}+Ch^2+\varepsilon h\Phi_1}{\varepsilon+hK_1+hK_2}. \tag{2.8}$$

It follows directly from (2.6) that

$$\left|\tilde{\beta}^{\varepsilon,h}_{0}\right| \le \frac{2\tilde{\delta} + ch^2 + \varepsilon\,|\psi(s_1)|}{\varepsilon + hK_1 + hK_2} \tag{2.9}$$

If $\psi(s_0) = 0$, then

$$\left|\tilde{\beta}^{\varepsilon,h}_{0}\right| \le \frac{2\tilde{\delta} + Ch^2 + \varepsilon h\Phi}{\varepsilon + hK_1 + hK_2}, \quad \Phi = \|\psi(s)\|_C \tag{2.9'}$$

We denote by μ_0 and μ_1 the right sides of inequalities (2.9) and (2.9′) respectively. Let's find an estimate of $\left|\tilde{\beta}^{\varepsilon,h}_{1}\right|$. Putting (2.8) in $i = 2$ we find:
$\left|\tilde{\beta}^{\varepsilon,h}_{1}\right| \le \mu_0\left(1 + \frac{\varepsilon+h^2L_1+h^2L_2}{\varepsilon+hK_1+hK_2}\right)$ if $\psi(s_0) = 0$, and $\left|\tilde{\beta}^{\varepsilon,h}_{1}\right| \le \mu_0 + \frac{\varepsilon+h^2L_1+h^2L_2}{\varepsilon+hK_1+hK_2}\mu_1$, otherwise, since $\psi(t_0) = 0$.
Let us consider the difference equation

$$\mu_i = \frac{\varepsilon + h^2L_1 + h^2L_2}{\varepsilon + hK_1 + hK_2}\mu_{i-1} + \frac{\varepsilon + h^2L_1 + h^2L_2}{\varepsilon + hK_1 + hK_2}\sum_{j=1}^{i-2}\mu_j + \mu_0, \quad i = \overline{1,n},\ \mu_1 = \mu_0. \tag{2.10}$$

It is obvious that

$$\left|\tilde{\beta}^{\varepsilon,h}_{i-1}\right| \le \mu_i, \quad i = \overline{1,n}. \tag{2.11}$$

Finding the difference $\mu_i - \mu_{i-1}$, and reducing (2.10) to the Cauchy problem for a homogeneous difference equation of the second order, we obtain:

$$\mu_i - \left(\frac{1 + \varepsilon + h^2L_1 + h^2L_2}{\varepsilon + hK_1 + hK_2}\right)\mu_{i-1} + \frac{\varepsilon}{\varepsilon + hK_1 + hK_2}\mu_{i-2} = 0, \quad i = \overline{3,n}, \tag{2.12}$$

$$\mu_1 = \mu_0 \quad \mu_2 = \left(1 + \frac{\varepsilon + h^2L_1 + h^2L_2}{\varepsilon + hK_1 + hK_2}\right)\mu_0. \tag{2.13}$$

The solution (2.12), (2.13) is well known and has the form:

$$\mu_i = l_1\lambda_1^{i-1} + l_2\lambda_2^{i-1}, \quad i = \overline{1,n}, \tag{2.14}$$

where λ_1, λ_2 - are the roots of the characteristic equation
$\lambda^2 - \left(1 + \frac{\varepsilon+h^2L_1+h^2L_2}{\varepsilon+hK_1+hK_2}\right)\lambda + \frac{\varepsilon}{\varepsilon+hK_1+hK_2} = 0$, and $l_1 = \frac{\mu_1\lambda_2-\mu_2}{\lambda_2-\lambda_1}, \quad l_2 = \frac{\mu_2-\mu_1\lambda_1}{\lambda_2-\lambda_1}$.
Evaluation gives us the following:

$$\lambda_{1,2} = \frac{2\varepsilon + h(K_1+K_2) \pm [h(K_1+K_2) + h^2(L_1+L_2)\eta(\varepsilon,h)]}{2(\varepsilon + h(K_1+K_2))}, \tag{2.15}$$

here

$$\eta(\varepsilon,h) = \sqrt{1 + 4\varepsilon(L_1+L_2)[(K_1+K_2) + h((L_1+L_2))]}, \tag{2.16}$$

$$l_1 = \frac{\mu_0}{2}\left(1 + \frac{2\varepsilon + h^2(L_1+L_2)}{[h(K_1+K_2) + h^2(L_1+L_2)\eta(\varepsilon,h)]} + \frac{1}{\eta(\varepsilon,h)}\right), \tag{2.17}$$

$$l_2 = \frac{\mu_0}{2}\left(1 - \frac{2\varepsilon + h^2(L_1+L_2)}{[h(K_1+K_2) + h^2(L_1+L_2)\eta(\varepsilon,h)]} - \frac{1}{\eta(\varepsilon,h)}\right). \tag{2.18}$$

Formulas (2.15)-(2.18) completely determine the solution to the problem (2.13), (2.14). Since $\lambda_2 \geq 0 \quad l_2 \leq 0 \quad \forall\, \alpha \geq 0$, from (2.14), we have

$$\mu_i \leq l_1\lambda_1^{i-1}, \quad i = \overline{1,n.} \tag{2.19}$$

It is obvious that $\eta(\varepsilon,h) = 1 + \mathrm{O}(\varepsilon), \quad \forall\, h > 0$, and it follows from (2.15) $\lambda_1 = 1 + \mathrm{O}(h)$, and (2.17) gives $l_1 \leq \mu_0 = const$. Thus from (2.19), we get:
$\mu_i \leq \mu_0 = const,\ i = \overline{1,n}$.
Then taking into account that $\mu_0 = \frac{2\tilde{\delta}+Ch^2+\varepsilon h\Phi}{\varepsilon+h(K_1+K_2)}$, we arrive to inequality $\mu_i \leq \frac{c_1\tilde{\delta}+c_2h^2+c_3\varepsilon h}{\varepsilon+h(K_1+K_2)}$, $i = \overline{1,n}, \quad c_1, c_2, c_3 = const$. By virtue of (11) we obtain

$$\left|\tilde{\beta}_{i-1}^{\varepsilon,h}\right| \leq \frac{c_1\tilde{\delta} + c_2h^2 + c_3\varepsilon h}{\varepsilon + h(K_1+K_2)}, \quad i = \overline{1,n}. \tag{2.20}$$

Thus, the desired error estimate of the regularized framework of the approximate solution of the equation is obtained. We call such dependencies quasi-optimally consistent $h(\tilde{\delta})$ and $\varepsilon(\tilde{\delta})$, which maximize the rate at which the right-hand sides of (2.18) tend to zero when $\tilde{\delta} \to 0$.

It is easy to see that in the case $\psi(0) = 0$: $h_{k.o.s.}(\tilde{\delta}) \approx \tilde{\delta}, \quad \varepsilon_{k.o.s.}(\tilde{\delta}) \approx \tilde{\delta}^{\frac{1}{2}}$, at the same time $\left\|\tilde{\beta}^{\varepsilon_{k.o.s.},h_{k.o.s.}}\right\|_{C_{n\ k.o.s.}} = \mathrm{O}(\tilde{\delta}^{\frac{1}{2}})$

Theorem 2.1. *For ε-regularized framework of an approximate solution of equation (2.1) satisfying SLAE (2.3), if with $\varepsilon \to 0$, the following assessment is valid:* $\left|\psi(s_{i-1}) - \psi_{i-1}^{\varepsilon(\tilde{\delta}),h(\tilde{\delta})}\right| = \mathrm{O}(\tilde{\delta}^{\frac{1}{2}}),\ i = \overline{1,n}$.

Let's consider the equation $\delta z_i + \sum\limits_{j=1}^{i} z_i h = \psi_{\varepsilon i}$, and it is given that $z(0) = 0$ and $z(t)$ - the exact solution of equation (1), then the following estimate is valid [2]: $\left\|z(t) - \tilde{z}^{\varepsilon,\delta}(t)\right\|_{C_n} \leq d_1\delta + d_2\frac{\varepsilon}{\delta} \leq \mathrm{O}(\varepsilon^{\frac{1}{2}})$, where $d_1, d_2 = const$, $\delta(\varepsilon) \approx \varepsilon^{\frac{1}{2}}$.

3. Develop a regularization-numerical algorithm

A regularization-numerical algorithm can also be introduced based on the direct system (2.2):

$$\begin{cases} \sum\limits_{j=1}^{i} K_{0j}\psi_j h + \sum\limits_{j=1}^{i} N_{ij}\psi_j h = f_i + K_0, \\ \sum\limits_{j=1}^{i} z_j h = \psi_i, \quad i = \overline{1,n}. \end{cases} \tag{3.1}$$

Let us denote by η_i and R_i, respectively, the error vector of the method and the error of the formula of right rectangles on the segment $[0, x_i)$ $\eta_i = \psi_i - \psi_{\varepsilon,i}$,
$R_i = \int\limits_0^{x_i} N(x_{i,j}, s)\psi(s)ds - h\sum\limits_{j=1}^{i} N_{i,j}\psi_j$. By applying the right rectangle formula and introducing the substitution $\eta_i = \psi_i^T - \psi_i$, we get:
$h\sum\limits_{j=1}^{i}(K_{0i,j}\eta_j + N_{ij}(\eta_i + \psi_i^T)) = R_i^* - \sum\limits_{j=1}^{n}[K_{i,j}^0(z_j) - K_{i,j}^0(z_{\varepsilon,j})]$,
where $\eta_0 = z_{\varepsilon,\eta} - z(0) = z_{\xi,\eta}$, $R_i^* = R_i + R_i'$, $R_i' = \int\limits_0^{x_0} K^0(x_i, s)\psi(s)ds - h\sum\limits_{i=1}^{n} K_{ij}^0\psi_i$.
The following estimates take place:
$|\eta_i| \le \frac{h}{h|K_{0i,j}|}\sum\limits_{j=1}^{i-1}[|K_{0i,j} - K_{0i-1,j}|\,|\eta_j| + \left|N_{i,j}(\eta_j + \psi_j^T) - N_{i-1,j}(\eta_j + y_j^T) + N_{i-1,j}(y_j^T)\right|] +$
$+\frac{1}{h|K_{0i,j}|}[\left|R_i^* - R_{i-1}^*\right| + h\sum\limits_{j=1}^{n}\left|K_{i,j}^0(z_j) - K_{i,j}^0(z_{\varepsilon,j}) - K_{i-1,j}^0(z_j) + K_{i-1,j}^0(z_{\varepsilon,j})\right|]$.
Let $L_m = \max\limits_{[0,b]\times R'}\left|\frac{\partial^2}{\partial x\partial s}N(x,s)\right|$, $L_k = \max\limits_{[0,b]}|K_x(x,s)|$.
Then we get $|\eta_i| \le \frac{L_k+L_m}{|K_{i,i}|}h\sum\limits_{j=1}^{n}|\eta_j| + \frac{L_k+L_m}{|K_{i,i}|}h\sum\limits_{j=1}^{n}\left|K_{i,j}^0(z_j) - K_{i,j}^0(z_{\varepsilon,j})\right| + \frac{1}{h|K_{i,i}|}\left|R_i^* - R_{i-1}^*\right|$.
Assessing: $\left|R_i^* - R_{i-1}^*\right| \le \frac{h^2 b}{2}\max\limits_{[0,b]\times R'}|(N_x'(x,s)\psi(s))_s'| + \frac{h^2}{2}\max\limits_{[0,b]\times R'}|(N(x,s)\psi(s))_s'| \le ch^2$
we have: $|\eta_i| \le c_1 h\sum\limits_{j=1}^{i-1}|\eta_i| + c_1 h\sum\limits_{j=1}^{n}\left|K_{i,j}^0(z_j) - K_{i,j}^0(z_{\varepsilon,j})\right| + c_2 h \le c_1 h\sum\limits_{j=1}^{i-1}|\eta_i| + Ph$,
where $c = \frac{h^2 b}{2}\max\limits_{[0,b]\times R'}|(N_x'(x,s)\psi(s))_s'| + \max\limits_{[0,b]\times R'}|N(x,s)\psi(s))_s'|$ $c_2 = \frac{c}{|K_{i,j}|}$, $c_1 = \frac{L_k+L_m}{|K_{i,i}|}$,
$p = c_1\sum\limits_{j=1}^{n}\left|K_{i,j}^0(z_j) - K_{i,j}^0(z_{\varepsilon,j})\right| + c_2$, $0 < c_2 = const$.
Arising in the study of numerical methods, they are c, m and i, usually interconnected. More precisely $c = c_1 h^\lambda$, $m = m_1 h$, $ih \le d$, where h is a grid step;c_1, λ, m_1, d - are independent on h constants; $u_i \le c_1^* h^\lambda (1+h)^{dh-1} \le c_1^* h^\lambda e^{m_1 d}$, u_i - the error modulus of the grid solution at the i-th grid node, we obtain:$|\eta_i| \le hPe^{c_1 b}$ where $c_1^* = P = c_1 b\sum\limits_{j=1}^{n}\left|K_{i,j}^0(z_j) - K_{i,j}^0(z_{\varepsilon,j})\right| + c_2, \lambda = 1$, $m_1 = c_1$, $d = b$. The convergence of order h, of this quadrature method has been proved.

4. Main Results

Implementing the Algorithm with NumPy.The NumPy library provides efficient work with arrays and matrices, significantly reducing computation time compared to traditional

loop implementations. The algorithm's code is provided below.

```
import numpy as np

def solve_volterra(n=100,
epsilon=0.005):

h = 1.0 / n
x = np.linspace(0, 1, n)
z = np.zeros(n)

def kernel(xi, tj):
return 1 + xi - tj

def f(x):
return x**2 / 2 + x**3 / 6 + epsilon*x

for i in range(n):
s = 0.0

for j in range(i):
s += kernel(x[i], x[j]) * z[j]

z[i] = (f(x[i]) - h*s) /
(epsilon + h*kernel(x[i], x[i]))

return x, z
```

The algorithm implements the proposed regularized scheme and calculates an approximate solution at the grid nodes.
A model solution was used to test the algorithm $z(x) = 1$.
The calculations were performed with different values of grid parameters and regularization.

n	h	Error
33	0.0303	0.0410
66	0.0151	0.0204
128	0.0078	0.0098
256	0.0039	0.0047

Table 1 – Dependence of error on the number of nodes

A stable convergence of the numerical solution is observed with decreasing grid step.

5. Conclusion

The obtained results confirm the effectiveness of regularization. A software implementation of the regularization-numerical method for solving the Volterra integral equation of the

ε	Error
10^{-1}	0.092
10^{-2}	0.041
10^{-3}	0.018
10^{-4}	0.011

Table 2 – Effect of the regularization parameter

first kind was developed using Python and the NumPy library. An analysis of the algorithm's stability, convergence, and computational complexity was conducted. The results of computational experiments demonstrated the effectiveness of the proposed approach and its applicability to a wide class of inverse problems in mathematical physics.

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Uniform Paracompactness and it's Exponential Spaces

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Abstract

In this paper we establish that the space X is (strongly) uniformly A-paracompact ((strongly) uniformly R-paracompact, strongly uniformly u-paracompact, weakly uniformly u-paracompact, (strongly) uniformly τ-star) if and only if $(\exp_c X, \exp_c U)$ is so. Further, we prove that if $f : (X, U) \to (Y, V)$ is a τ-bounded ((strongly) uniformly τ-star) mapping, so is $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$.

Keywords: uniform space, uniform paracompactness, strong uniform paracompactness, uniformly continuous mapping, exponential space, exponential mapping.
2020 MSC: 54E15.

1. Introduction

Recently, the theory of exponential spaces of uniform spaces and their mappings has been intensively developed. The exponential spaces of topological and uniform spaces have been studied in [1], [2], [5], [14] – [19].

We shall need a little of the terminology of the theory of quasi-ordered sets. Moreover, it will be convenient to use slightly nonstandard terminology.

A set S is said to be quasi-ordered by a relation "$<$" if "$<$" is transitive. A subset Q of S is cofinal in S if for each element s of S there exists an element q of Q such that $s < q$. Q is residual in S if whenever $q \in Q$ and $r < q$ in S, $r \in Q$. Q is antiresidual if for $q \in Q$ and $r > q$ in S, r is in Q. The quasi-ordered set S is directed if for any p and q in S there exists s in S satisfying $p < s$ and $q < s$. Such an s may be called a common successor of p and q. One important example of a quasi-ordered set is any family S of subsets of a given set X, ordered by the inclusion "$\subset$" [6].

For coverings α and β of a set X, we have: $\alpha \wedge \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. $\alpha(x) = \cup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(H) = \cup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha : A \cap H \neq \emptyset\}$, $H \subset X$. A covering α is called a refinement of a covering β if each $A \in \alpha$ is a subset of at least one $B \in \beta$. We write $\alpha \succ \beta$, and note that "$\succ$" is a quasi-ordering. A covering α is a star refinement of a covering β if for each $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(x) \subset B$ and we write $\alpha \triangleright \beta$, and note that "$\triangleright$" is a quasi-ordering. A covering α is a strongly star refinement of a covering β if for each $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(A) \subset B$, and one writes $\alpha* \succ \beta$. The relation "$* \succ$" is again a quasi-ordering. For the covering α of a set X let $\alpha^{\angle} = \{\cup\alpha_0 : \alpha_0 \subset \alpha, \alpha_0 \ \ is \ \ finite\}$. The covering α will be called finitely-additive if $\alpha^{\angle} = \alpha$.

A uniformity on a nonempty set X is a family U of coverings of X which satisfies the following conditions:

(U1) if $\alpha \in U$ and if β is a covering of X such that $\alpha \succ \beta$, then $\beta \in U$;

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(U2) if $\alpha_1, \alpha_2 \in U$, then there exists $\alpha \in U$ such that $\alpha \succ \alpha_1$ and $\alpha \succ \alpha_2$;

(U3) if $\alpha \in U$, then there exists $\beta \in U$ such that $\beta* \succ \alpha$;

(U4) for any two distinct points $x, y \in X$ there is a uniform covering $\alpha \in U$ such that $\alpha(x)$ and $\alpha(y)$ are disjoint [2], [4], [10], [11], [13].

The elements of U are called uniform coverings. A uniform spaces (X, U) is a set X with a uniformity U on X. The family U satisfying the conditions $(U1) - (U3)$ is called pseudouniformity on X and the pair (X, U) is said to be a pseudouniform space [2].

The uniform topology of a uniform space X is defined as follows. A subset O of X is neighborhood of a point x of O if for some uniform covering α, O contains $\alpha(x)$. O is open if it is a neighborhood of each of its points. Every uniform space is a completely regular Hausdorff space in the uniform topology [6].

The uniformities or preuniformities on any set form a partially ordered set under inclusion. The preuniformities, like the topologies, form a complete lattice but not lattice. Evidently, any preuniformity which contains a uniformity is a uniformity. As with topologies, a preuniformity U containing a preuniformity V is said to be finer than V. The usage of the terms "strong" and "weak" is not standardized, and we shall avoid them as far as possible. The finest uniformity on the completely regular Hausdorff space X is called the universal uniformity on the space X and is denoted by U_X. A basis for a uniformity U is a filter base for U considered as a filter of coverings. A subbasis for a uniformity or preuniformity is a family of coverings whose finite intersections form a basis. A family W of coverings which satisfies condition $(U3)$ is called a normal family. It is convenient and customary to use the term normal sequence for something more special than a sequence which is a normal family: specifically, for a sequence of coverings α_n such that $\alpha_{n+1}* \succ \alpha_n$ for each n [2], [6].

A collection α of sets in a uniform space (X, U) is called uniformly locally finite if there is a uniform covering $\beta \in U$ each of whose elements meets only finitely many elements of α [6]. A covering α of a uniform space (X, U) is called uniformly star-finite if there is a uniform covering $\beta \in U$ such that for each $B \in \beta$ the set $\alpha(B)$ meets only finitely many elements of α [10]. A covering α of a uniform space (X, U) is called τ-star (strongly τ-star), if $|St(\alpha, x)| \leq \tau$ for any $x \in X$ ($|St(\alpha, A)| \leq \tau$ for any $A \in \alpha$)[10]. A uniform space (X, U) is called (strongly, weakly) uniformly u-paracompact if for each uniform covering $\alpha \in U$ there exists a (star-finite, point-finite) locally finite uniform covering $\beta \in U$, such that $\beta \succ \alpha$ [10], [11]. A uniform space (X, U) is called (strongly) uniformly A-paracompact if every finitely-additive open covering has a (star finite) locally finite uniform refinement [11]; a uniform space (X, U) is called (strongly) uniformly R-paracompact if every open covering has a (uniformly star-finite) uniformly locally finite open refinement [2], [3], [9], [10], [11]; a uniform space (X, U) is called precompact if for each uniform covering $\alpha \in U$ there exists a finite uniform covering $\beta \in U$, such that $\beta \succ \alpha$ [2], [3], [8]; a uniform space (X, U) is called pre-Lindelöf if for each uniform covering $\alpha \in U$ there exists a countable uniform covering $\beta \in U$, such that $\beta \succ \alpha$ [3], [11], [13]; a uniform space (X, U) is called τ-bounded if for each uniform covering $\alpha \in U$ there exists a uniform covering $\beta \in U$ cardinality $\leq \tau$, such that $\beta \succ \alpha$ [2], [3], [10]; a uniform space (X, U) is called (strongly) uniformly τ-star if for each uniform covering $\alpha \in U$ there exists a (strongly) τ-star uniform covering $\beta \in U$ such that $\beta \succ \alpha$ [10].

Let $f : X \to Y$ be a mapping. If α and β are the coverings of X and Y, respectively, then $f\alpha = \{fA : A \in \alpha\}$ and $f^{-1}\beta = \{f^{-1}B : B \in \beta\}$ are coverings of Y and X, respectively. A mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called uniformly continuous if for any $\beta \in V$ there exists $\alpha \in U$ such that $f\alpha \succ \beta$. It is easy to see that a mapping f between uniform spaces (X, U) and (Y, V) is uniformly continuous

if and only if $f^{-1}\beta \in U$ for any $\beta \in V$. A bijective mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) onto a uniform space (Y, V) is called uniform homomorphism or a uniform isomorphism if both mappings $f : (X, U) \to (Y, V)$ and $f^{-1} : (Y, V) \to (X, U)$ are uniformly continuous. A uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called a precompact if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and finite uniform covering $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [2], [10]; a uniformly continuous mapping f between uniform spaces (X, U) and (Y, V) is called uniformly perfect if it is both precompact and perfect [2], [10]; a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called pre-Lindelöf if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and countable uniform covering $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [10]; a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called τ-bounded if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and a uniform covering $\gamma \in U$ cardinality $\leq \tau$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [10]; a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called (strongly) uniformly τ-star if for each uniform covering $\alpha \in U$ there exist a uniform covering $\beta \in V$ and a (strongly) τ-star uniform covering $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [10].

Let $\exp_c X$ be the set of all non-empty compact subsets of the space (X, τ_U). For each $\alpha \in U$ we put $K(\alpha) = \{\langle\alpha'\rangle : \alpha' \subset \alpha \; is \; finite\}$, where $\langle\alpha'\rangle = \{K \in \exp_c X : K \subset \cup\alpha', K \cap A \neq \emptyset \; \forall A \in \alpha'\}$. A uniform space $(\exp_c X, \exp_c U)$ is called the exponential space of compact subsets of a uniform space (X, U) with the Vietoris topology and the uniformity $\exp_c U$ is called Hausdorff uniformity on $\exp_c X$ [2], [4], [16].

By a uniform space we shall mean a Hausdorff uniform space and by a mapping – a uniformly continuous mapping between two uniform spaces [1] – [13], [16], [19].

2. Main Results

Let (X, U) be an arbitrary uniform space and $(\exp_c X, \exp_c U)$ be its exponential space.

Lemma 2.1. Let α and β coverings of a uniform space (X, U). Then $K(\alpha)$ and $K(\beta)$ are coverings of the space $(\exp_c X, \exp_c U)$. If $\alpha \succ \beta$, then $K(\alpha) \succ K(\beta)$.

Proof. Let α and β be coverings of (X, U). Let $K \in \exp_c X$ be an arbitrary element. Then there exists a finite subfamily of $\alpha' = \{A_1, A_2, \ldots, A_n\}$ of the covering α such that $K \subset \cup\alpha'$ and $K \cap A_i \neq \emptyset$ for all $i = 1, 2, \ldots, n$. Consequently, $K \in \langle\alpha'\rangle$ and $\langle\alpha'\rangle \in K(\alpha)$. This means that $K(\alpha)$ is a covering of the space $(\exp_c X, \exp_c U)$. Now let $\langle\alpha'\rangle \in K(\alpha)$ be an arbitrary element. We show that $\langle\alpha'\rangle \subset \langle\beta'\rangle$. Let $K \in \langle\alpha'\rangle$ be some element. Then $K \subset \cup\alpha'$ and $K \cap A_i \neq \emptyset$ for all $i = 1, 2, \ldots, n$. Since $\alpha \succ \beta$ for any $A_i \in \alpha'$ there exists $B_{A_i} \in \beta$ such that $A_i \subset B_{A_i}$. Let $\beta' = \{B_{A_1}, B_{A_2}, \ldots, B_{A_n}\}$. Note that $K \subset \cup\beta'$ and $K \cap B_{A_i} \neq \emptyset$ for all $i = 1, 2, \ldots, n$. It follows that $K \in \langle\beta'\rangle$, i.e. $\langle\beta'\rangle$ is the desired element of the covering $K(\beta)$. Therefore, $K(\alpha) \succ K(\beta)$. □

Lemma 2.2. Let α be a covering of the space (X, U). Put $\alpha_{\aleph_0} = \{\cup\alpha_0 : \alpha_0 \subset \alpha \text{ is finite subfamily}\}$. Then $K(\alpha_{\aleph_0})$ is a covering of the space $(\exp_c X, \exp_c U)$.

Proof. Let $K \in \exp_c X$ be an arbitrary element. Since $\alpha_{\aleph_0}$ is a covering and $K \subset X$ is compact, there exists a finite subfamily $\alpha'_{\aleph_0} = \{A_1^{\swarrow}, A_2^{\swarrow}, \ldots, A_n^{\swarrow}\} \subset \alpha_{\aleph_0}$, $A_j^{\swarrow} = \bigcup_{i=1}^{m} A_i$, $A_i \in \alpha$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, such that $K \subset \cup\alpha'_{\aleph_0}$ and $K \cap A_j^{\swarrow} \neq \emptyset$ for all $j = 1, 2, \ldots, n$. Therefore, $K \in \langle\alpha'_{\aleph_0}\rangle$ and $\langle\alpha'_{\aleph_0}\rangle \in K(\alpha_{\aleph_0})$. Hence, $K(\alpha_{\aleph_0})$ is a covering of the space $(\exp_c X, \exp_c U)$. □

Lemma 2.3. If β is a star-finite uniform covering of the space (X, U), then $K(\beta)$ is a star-finite uniform covering of the space $(\exp_c X, \exp_c U)$.

Proof. Let us show what $K(\beta)$ is a star-finite uniform covering. Let $\langle\beta'\rangle \in K(\beta)$ be an arbitrary element. Let $\langle\beta''\rangle \in K(\beta)$ be an element such that $\langle\beta'\rangle \cap \langle\beta''\rangle \neq \emptyset$, where $\beta' = \{B_1^{\beta'}, B_2^{\beta'}, \ldots, B_n^{\beta'}\}$, $\beta'' = \{B_1^{\beta''}, B_2^{\beta''}, \ldots, B_m^{\beta''}\}$. Then there exists $K \in \langle\beta'\rangle \cap \langle\beta''\rangle$ such that $K \subset (\cup\beta') \cap (\cup\beta'')$ and $K \cap B_i^{\beta'} \neq \emptyset$, $K \cap B_j^{\beta''} \neq \emptyset$, for all i, j, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Then for each i there exists $j(i)$ such that $B_i^{\beta'} \cap B_{j(i)}^{\beta''} \neq \emptyset$. Consequently, the number of such $\langle\gamma''\rangle$ intersecting $\langle\gamma'\rangle$ is finite. □

Theorem 2.1. *A uniform space (X, U) is uniformly A-paracompact if and only if $(\exp_c X, \exp_c U)$ is uniformly A-paracompact.*

Proof. Necessity. Let (X, U) be a uniformly A-paracompact space and $K(\alpha)$ be an arbitrary finitely additive open covering. Then for the finitely additive open covering α there exists a locally finite uniform covering $\gamma \in U$ such that $\gamma \succ \alpha$. According to Lemma 2.1. $K(\gamma) \succ K(\alpha)$. We show what $K(\gamma)$ is a locally finite uniform covering. Let $K \in \exp_c X$ be an arbitrary element. Then there exists $\langle\gamma'\rangle \in K(\gamma)$ such that $K \in \langle\gamma'\rangle$, i.e. $K \subset \cup\langle\gamma'\rangle$ and $K \cap G_i \neq \emptyset$, $G_i \in \gamma'$, $i = 1, 2, \ldots, n$. Since $\gamma \in U$ is a locally finite uniform covering, then for any point $x \in X$ there exists a neighborhood O_x such that $O_x \subset \cup_{j=1}^{k} G_j$. Let $\lambda = \{O_x : x \in X\}$. Then there exists a finite subfamily $\lambda' = \{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$ such that $K \subset \bigcup_{i=1}^{n} O_{x_i}$. It is clear that for any $x_i \in K$, $i = 1, 2, \ldots, n$, $O_{x_i} \subset \bigcup_{j=1}^{k} G_j^{x_i}$. Let $\gamma'_{x_i} = \{G_1^{x_i}, G_2^{x_i}, \ldots, G_k^{x_i}\}$, $i = 1, 2, \ldots, n$. It follows that $O = \bigcup_{i=1}^{n} O_{x_i} \subset \bigcup_{i=1}^{n} \langle\gamma'_{x_i}\rangle$. Therefore, $K(\gamma)$ is a locally finite uniform covering.

Sufficiency follows from the fact that every closed subspace of a uniformly A-paracompact space is a uniformly A-paracompact space. □

Theorem 2.2. *A uniform space (X, U) is strongly uniformly A-paracompact if and only if $(\exp_c X, \exp_c U)$ is strongly uniformly A-paracompact.*

Proof. Necessity. Let (X, U) be a strongly uniformly A-paracompact space and $K(\alpha) \in \exp_c U$ be an arbitrary uniform covering. Then there exists a star-finite uniform covering $\gamma \in U$ such that $\gamma \succ \alpha$. It follows from Lemma 2.1. that $K(\gamma) \succ K(\alpha)$. We show what $K(\gamma)$ is a star-finite uniform covering. The star finiteness of $K(\gamma)$ follows from Lemma 2.3. Consequently, $K(\gamma)$ is a star-finite uniform covering. Thus, $(\exp_c X, \exp_c U)$ is strongly uniformly A-paracompact.

Sufficiency follows from the fact that every closed subspace of a strongly uniformly A-paracompact space is strongly uniformly A-paracompact. □

Theorem 2.3. *A uniform space (X, U) is strongly uniformly u-paracompact if and only if $(\exp_c X, \exp_c U)$ is strongly uniformly u-paracompact.*

Proof. Necessity. Let (X, U) be a strongly uniformly u-paracompact space and $K(\alpha) \in \exp_c U$ be an arbitrary element. Then there exists a star-finite uniform covering $\beta \in U$ such that $\beta \succ \alpha$. From Lemma 2.1. it follows that $K(\beta) \succ K(\alpha)$. Therefore, $K(\beta) \in \exp_c U$. By Lemma 2.3. $K(\beta)$ is a star-finite covering. Hence, $(\exp_c X, \exp_c U)$ is strongly uniformly u-paracompact.

Sufficiency. It follows from the fact that every closed subspace of a strongly uniformly u-paracompact space is strongly uniformly u-paracompact. □

Theorem 2.4. *A uniform space (X, U) is weakly uniformly u-paracompact if and only if $(\exp_c X, \exp_c U)$ is weakly uniformly u-paracompact.*

Proof. Necessity. Let (X, U) be a weakly uniformly u-paracompact space and $K(\alpha) \in \exp_c U$ be an arbitrary uniform covering. Then there exists a point-finite uniform covering $\beta \in U$ such that $\beta \succ \alpha$. According to a statement of J. Isbell (see [6], Lemma 3, P. 124, or see [7], Statement 1.2., P. 109) there exists a uniformly locally finite uniform covering $\gamma \in U$ such that $\gamma \succ \beta$. From Lemma 2.1. it follows that $K(\gamma) \succ K(\alpha)$ and $K(\gamma) \in \exp_c U$. Now we show that $K(\gamma)$ is a point-finite uniform covering. Let $\lambda \in U$ be a uniform covering such that each of its elements intersects finitely many elements of the covering γ, i.e. $L \subset \bigcup_{i=1}^{n} G_i$, $L \in \lambda$, $G_i \in \gamma$, $i = 1, 2, \ldots, n$. Let $K \in \exp_c X$ be an arbitrary compact. Then there exists $\langle \lambda' \rangle \in K(\lambda)$ such that $K \in \langle \lambda' \rangle$ and $K \subset \cup\lambda'$, $K \cap L_i \neq \emptyset$, $L_i \in \lambda'$, $i = 1, 2, \ldots, n$. If $K \in \langle \gamma' \rangle$, then $K \cap G_i \neq \emptyset$, $G_i \in \gamma'$, $i = 1, 2, \ldots, n$. Then for each i there exists an element $L_i \in \lambda'$ such that $L_i \cap G_i \neq \emptyset$. But each element $L_i \in \lambda'$ intersects only finitely many elements of γ. Consequently, K contains at most finitely many elements of the covering $K(\gamma)$. Hence, $(\exp_c X, \exp_c U)$ is weakly uniformly u-paracompact.

Sufficiency follows from the fact that every closed subspace of weakly uniformly u-paracompact space is weakly uniformly u-paracompact. □

Theorem 2.5. *A uniform space (X, U) is strongly uniformly R-paracompact if and only if $(\exp_c X, \exp_c U)$ is strongly uniformly R-paracompact.*

Proof. Necessity. Let (X, U) be a strongly uniformly R-paracompact space and $K(\alpha)$ an arbitrary open covering of $(\exp_c X, \exp_c U)$. Then there exists a uniformly star-finite open covering β such that $K(\beta) \succ K(\alpha)$. We show that $K(\beta)$ is a uniformly star-finite open covering. For this it is sufficient to prove that for any $\langle \gamma' \rangle \in K(\gamma)$ the set $K(\beta)(\langle \gamma' \rangle)$ intersects only finitely many elements of the covering $K(\beta)$. Let $\langle \beta' \rangle \in K(\beta)$ be an element such that $K(\beta)(\langle \gamma' \rangle) \cap \langle \beta' \rangle \neq \emptyset$. Then there exists a compact $K \in K(\beta)(\langle \gamma' \rangle) \cap \langle \beta' \rangle$ such that $K \in \langle \beta'' \rangle \cap \langle \beta' \rangle$, $K \subset (\cup\beta'') \cap (\cup\beta'')$ and $K \cap B_i^{\beta''} \neq \emptyset$, $K \cap B_j^{\beta'} \neq \emptyset$, for all $i, j, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. Then for each i there exists a number $j(i)$ such that $B_i^{\beta''} \cap B_{j(i)}^{\beta'} \neq \emptyset$. Since $\langle \beta'' \rangle \cap \langle \gamma' \rangle \neq \emptyset$, then there exists $K' \in \langle \beta'' \rangle \cap \langle \gamma' \rangle$ such that $K' \subset (\cup\beta'') \cap (\cup\gamma')$ and $K' \cap B_i^{\beta''} \neq \emptyset$, $K' \cap G_{p'}^{\gamma'} \neq \emptyset$, for all $i, p, i = 1, 2, \ldots, n, p = 1, 2, \ldots, k$. Then for each i there exists $p(i)$ such that $B_i^{\beta''} \cap G_{p(i)}^{\gamma'} \neq \emptyset$. Therefore, $\beta(G_{p(i)}^{\gamma'}) \cap B_{j(i)}^{\beta'} \neq \emptyset$. But each $\beta(G_{p(i)}^{\gamma'})$ intersects only a finite number of elements β. It follows that the set of elements $\langle \beta' \rangle$ of the covering $K(\beta)$ intersects $K(\beta)\langle \gamma' \rangle$ only finitely many. Thus $K(\beta)$ is a uniformly star-finite open covering. Consequently, $(\exp_c X, \exp_c U)$ is strongly uniformly R-paracompact.

Sufficiency follows from the fact that every closed subspace of a strongly uniformly R-paracompact space is a strongly uniformly R-paracompact subspace. □

Theorem 2.6. *A uniform space (X, U) is strongly uniformly τ-star if and only if $(\exp_c X, \exp_c U)$ is strongly uniformly τ-star.*

Proof. Necessity. Let (X, U) be a strongly uniformly τ-star space. Choose an arbitrary element $K(\alpha)$ from $\exp_c U$. Then there exists a strongly τ-star uniform covering $\gamma \in U$ such that $\gamma \succ \alpha$. It follows from Lemma 2.1. that $K(\gamma) \succ K(\alpha)$. We show that $K(\gamma)$ is a strongly τ-star uniform covering. Let $\langle \gamma' \rangle \in K(\gamma)$ be an arbitrary element and $\langle \gamma'' \rangle \in K(\gamma)$ be an element such that $\langle \gamma' \rangle \cap \langle \gamma'' \rangle \neq \emptyset$, where $\gamma' = \{G_1^{\gamma'}, G_2^{\gamma'}, ..., G_n^{\gamma'}\}$, $\gamma'' = \{G_1^{\gamma''}, G_2^{\gamma''}, ..., G_m^{\gamma''}\}$. Then there exists $K \in \langle \gamma' \rangle \cap \langle \gamma'' \rangle$ such that $K \subset (\cup\gamma') \cap (\cup\gamma'')$. And $K \cap G_i^{\gamma'} \neq \emptyset, K \cap G_j^{\gamma'} \neq \emptyset$, for all $i, j, i = 1, 2, ..., n, j = 1, 2, ..., m$. Each element of γ' intersects no more than τ elements of the covering γ. Then the number of such $\langle \gamma'' \rangle$ intersecting with $\langle \gamma' \rangle$ is not greater than τ.

Therefore, $K(\beta)$ is a strongly τ-star uniform covering. Hence, $(\exp_c X, \exp_c U)$ is a strongly uniformly τ-star space.

Sufficiency. It follows from the fact that every closed subspace is strongly uniformly τ-star space is a strongly uniformly τ-star space.

A collection α of sets in a uniform space (X, U) is called uniformly τ-locally if there is a uniform covering $\beta \in U$ each of whose elements meets only τ elements of α. □

Theorem 2.7. *A uniform space (X, U) is uniformly τ-star if and only if $(\exp_c X, \exp_c U)$ is uniformly τ-star.*

Proof. Let (X, U) be a uniformly τ-star space and $K(\alpha) \in \exp_c U$ be an arbitrary uniform covering. Then there exists a τ-star uniform covering $\beta \in U$ such that $\beta \succ \alpha$. Let $\gamma \in U$ be a uniformly τ-locally uniform covering such that $\gamma \succ \beta$. Again, from Lemma 2.1. it follows that $K(\gamma) \succ K(\alpha)$ and $K(\gamma) \in \exp_c U$. Now we show what $K(\gamma)$ is a τ-star uniform covering. Let $\lambda \in U$ be a uniform covering such that each element of which is intersected by only τ elements of the covering γ. Let $K \in \exp_c X$ be an arbitrary compact. If $K \in \langle\gamma'\rangle$, then $K \cap G_i \neq \emptyset$, $G_i \in \gamma'$, $i = 1, 2, \ldots, n$. Then for each i there exists an element $L_i \in \lambda'$ such that $L_i \cap G_i \neq \emptyset$. But each element $L_i \in \lambda'$ intersected at most τ elements of the covering γ. Consequently, K contains at most τ elements of the covering $K(\gamma)$. Hence, $(\exp_c X, \exp_c U)$ is uniformly τ-star space.

Sufficiency. It follows from the fact that any closed subspace of uniformly τ-star space is uniformly τ-star space.

Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping between uniform spaces (X, U) and (Y, V), $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ be a uniformly continuous mapping between exponential spaces $(\exp_c X, \exp_c U)$ and $(\exp_c Y, \exp_c V)$. □

Theorem 2.8. *If a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is uniformly τ-star, so $\exp_c(f)(K) = fK$.*

Proof. Let $f : (X, U) \to (Y, V)$ be a uniformly τ-star mapping and $K(\alpha) \in \exp_c U$ be some element. Then there exist a uniform covering $\beta \in V$ and a τ-star uniform covering $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$. It is clear that $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. Now we will show that $K(\gamma)$ is a τ-star uniform covering. Since the covering $\gamma \in U$ is a τ-locally uniform covering there exists a uniform covering λ such that it every element intersects only τ many elements of the covering γ. i.e., $L \subset \bigcup_{j=1}^{k} G_j$, $L \in \lambda$. If $K \in \exp_c X$ is an arbitrary element. Then there exists an element $\langle\gamma'\rangle \in K(\gamma)$ such that $K \in \langle\gamma'\rangle$ i.e. $K \subset \bigcup\langle\gamma'\rangle$ and $K \cap G_i \neq \emptyset$, $G_i \in \gamma'$, $i = 1, 2, \ldots, n$ and there exists a family $\lambda' = \{L_1, L_2, \ldots, L_n\}$ such that $K \subset \bigcup_{i=1}^{n} L_n$.

Consequently, for each i there is an element L_i of λ' with $L_i \cap G_i \neq \emptyset$. But each L_i meets only finitely many elements of the covering γ. Hence K belongs to not more than finitely many elements of $K(\gamma)$. □

Proposition 2.1. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping between uniform spaces (X, U) and (Y, V). If $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ is its exponential mapping, $\exp_c Y = \{K\}$ and $\exp_c(f)$ are uniformly τ-star, then $(\exp_c X, \exp_c U)$ is also uniformly τ-star.

Proof. Let $K(\alpha) \in \exp_c U$ be an arbitrary covering. There exists $K(\beta) \in \exp_c V$ and a τ-star covering $K(\gamma) \in \exp_c U$ such that $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. Since $\exp_c Y = \{K\}$, then $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) = K(\gamma) \succ K(\alpha)$, where $K(\gamma) = \{\langle\gamma'_1\rangle, \langle\gamma'_2\rangle, \ldots, \langle\gamma'_n\rangle\}$. Therefore, $(\exp_c X, \exp_c U)$ is uniformly τ-star. □

Proposition 2.2. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform spaces (Y, V) and $\exp_c f := (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ be its exponential mapping. If $(\exp_c X, \exp_c U)$ is uniformly τ-star space, then $\exp_c(f)$ is uniformly τ-star mapping.

Proof. Let $K(\alpha) \in \exp_c U$ be an arbitrary covering. Then by the uniformly τ-starredness of $(\exp_c X, \exp_c U)$ there exists a τ-star uniform covering $K(\gamma) \in \exp_c U$ such that $K(\gamma) \succ K(\alpha)$. Then for any uniform covering $K(\beta) \in \exp_c V$, we have $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. Thus, $\exp_c f$ is uniformly τ-star mapping. □

Proposition 2.3. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V) and $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$ be its exponential mapping. If $\exp_c f$ and $(\exp_c Y, \exp_c V)$ are uniformly τ-star, then $(\exp_c X, \exp_c U)$ is also uniformly τ-star.

Proof. Let $K(\alpha) \in \exp_c U$ be an arbitrary covering. There exists $K(\beta) \in \exp_c V$ and a τ-star uniform covering $K(\gamma) \in \exp_c U$ such that $\exp_c(f)^{-1}(K(\beta)) \wedge K(\gamma) \succ K(\alpha)$. By virtue of the uniformly τ-starredness of the space $(\exp_c Y, \exp_c V)$ there exists a τ-star uniform covering $K(\lambda) \in \exp_c V$ such that $K(\lambda) \succ K(\beta)$. It is easy to see that $\exp_c(f)^{-1}K(\lambda) \wedge K(\gamma) \succ \exp_c(f)^{-1}K(\beta) \wedge K(\gamma)$. We show that the covering $\exp_c(f)^{-1}K(\lambda)$ is a τ-star uniform covering. Since the mapping $\exp_c f$ is uniformly continuous, it follows that $\exp_c(f)^{-1}K(\lambda) \in \exp_c U$. Let $K \in \exp_c X$ be an arbitrary element. Since $\exp_c(f)(K) = fK$ then fK contains at most τ many elements from the covering $K(\lambda) \cup \langle\lambda_i'\rangle \ni fK$, where $\langle\lambda_i'\rangle \in K(\lambda)$ $i = 1, 2, \ldots, n$. It follows that $\bigcup_{i=1}^{n} \exp_c(f)^{-1}\langle\lambda_i'\rangle \ni K$. So, $K(\gamma)$ is a τ-star uniform covering of the space $(\exp_c X, \exp_c U)$. Therefore the space $(\exp_c X, \exp_c U)$ is uniformly τ-star. □

Theorem 2.9. *If a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is strongly uniformly τ-star, so is $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$.*

Proof. Let $f : (X, U) \to (Y, V)$ be a strongly uniformly τ-star mapping and $K(\alpha) \in \exp_c U$ be some element. Then there exist a uniform covering $\beta \in V$ and a strongly τ-star uniform covering $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$. It has been shown above that $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. We prove that $K(\gamma)$ is a strongly τ-star uniform covering. Let $\langle\gamma'\rangle \in K(\gamma)$ be an arbitrary element. Let $\langle\gamma''\rangle \in K(\gamma)$ be such an element that $\langle\gamma'\rangle \cap \langle\gamma''\rangle \neq \emptyset$, where $\gamma' = \{G_1^{\gamma'}, G_2^{\gamma'}, ..., G_n^{\gamma'}\}, \gamma'' = \{G_1^{\gamma''}, G_2^{\gamma''}, ..., G_m^{\gamma''}\}$. Then there exists $K \in \langle\gamma'\rangle \cap \langle\gamma''\rangle$ such that $K \subset (\cup\gamma') \cap (\cup\gamma'')$ and $K \cap G_i^{\gamma'} \neq \emptyset, K \cap G_j^{\gamma''} \neq \emptyset$ for all $i, j, i = 1, 2, ..., n, j = 1, 2, ..., m$. Thus for each i there is $j(i)$ with $G_i^{\gamma'} \cap G_{j(i)}^{\gamma''} \neq \emptyset$. Consequently, the number of all $\langle\gamma''\rangle$ meeting $\langle\gamma'\rangle$ is τ. □

Proposition 2.4. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping between uniform spaces (X, U) and (Y, V). If $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ is its exponential mapping, $\exp_c Y = \{K\}$ and $\exp_c f$ are strongly uniformly τ-star, then $(\exp_c X, \exp_c U)$ is also strongly uniformly τ-star.

Proof. Let $K(\alpha) \in \exp_c U$ be an arbitrary covering. There exist $K(\beta) \in \exp_c V$ and a strongly τ-star covering $K(\gamma) \in \exp_c U$ such that $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. Since $\exp_c Y = \{K\}$, we have $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) = K(\gamma) \succ K(\alpha)$, where $K(\gamma) = \{\langle\gamma_1'\rangle, \langle\gamma_2'\rangle, \ldots, \langle\gamma_n'\rangle\}$. Therefore, $(\exp_c X, \exp_c U)$ is strongly uniformly τ-star. □

Proposition 2.5. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping between uniform spaces (X, U) and (Y, V). Let $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ be its exponential mapping. If $(\exp_c X, \exp_c U)$ is strongly uniformly τ-star, then $\exp_c(f)$ is strongly uniformly τ-star.

Proof. Let $K(\alpha) \in \exp_c U$ be some covering. Then by the strong uniform τ-starredness of $(\exp_c X, \exp_c U)$ there exists a strongly τ-star uniform covering $K(\gamma) \in \exp_c U$ such that $K(\gamma) \succ K(\alpha)$. Then, for any $K(\beta) \in \exp_c V$, $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$ holds. Thus, $\exp_c f$ is strongly uniformly τ-star. □

Proposition 2.6. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V) and $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ be its exponential mapping. If $\exp_c f$ and $(\exp_c Y, \exp_c V)$ are strongly uniformly τ-star, then $(\exp_c X, \exp_c U)$ is also strongly uniformly τ-star.

Proof. Let $K(\alpha) \in \exp_c U$ be an arbitrary covering. There exist $K(\beta) \in \exp_c V$ and a strongly τ-star uniform covering $K(\gamma) \in \exp_c U$ such that $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. Then there exists a strongly τ-star uniform covering $K(\lambda) \in \exp_c V$ such that $K(\lambda) \succ K(\beta)$. It is easy to see that $\exp_c(f)^{-1}K(\lambda) \wedge K(\gamma) \succ \exp_c(f)^{-1}K(\beta) \wedge K(\gamma)$. We show that $\exp_c(f)^{-1}K(\lambda)$ is a strongly τ-star uniform covering. By uniform continuity, we have that $\exp_c(f)^{-1}K(\lambda) \in \exp_c U$. Let $K \in \exp_c X$ be an element such that $\exp_c(f)(K) = fK$ and $fK \in \langle\lambda'\rangle \cap \langle\lambda''\rangle$, where $\lambda' = \{L_1^{\gamma'}, L_2^{\gamma'}, ..., L_n^{\gamma'}\}$, $\lambda'' = \{L_1^{\gamma''}, L_2^{\gamma''}, ..., L_m^{\gamma''}\}$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$. Then $fK \subset (\cup\lambda') \cap (\cup\lambda'')$ and $fK \cap L_i^{\gamma'} \neq \emptyset$, $fK \cap L_j^{\gamma''} \neq \emptyset$, for all i, j, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$. It follows that $fK \subset (\cup f^{-1}\lambda') \cap (\cup f^{-1}\lambda'')$ and $K \cap f^{-1}L_i^{\gamma'} \neq \emptyset$, $K \cap L_j^{\gamma''} \neq \emptyset$, for all i, j, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, where $f^{-1}\lambda' = \{f^{-1}L_1^{\gamma'}, f^{-1}L_2^{\gamma'}, ..., f^{-1}L_n^{\gamma'}\}$, $f^{-1}\lambda'' = \{f^{-1}L_1^{\gamma''}, f^{-1}L_2^{\gamma''}, ..., f^{-1}L_m^{\gamma''}\}$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$. Consequently, every element of the covering $\exp_c(f)^{-1}K(\lambda)$ intersects only τ many elements of the covering $\exp_c(f)^{-1}K(\lambda)$. Thus, $\exp_c(f)^{-1}K(\lambda)$ is a strongly τ-star uniform covering of the space $(\exp_c X, \exp_c U)$. Therefore, $(\exp_c X, \exp_c U)$ is a strongly uniformly τ-star space. □

Theorem 2.10. *For a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) the following are equivalent:*

1. *$f : (X, U) \to (Y, V)$ is τ-bounded;*
2. *$\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$ is τ-bounded.*

Proof. 1) ⇒ 2). Let $f : (X, U) \to (Y, V)$ be a τ-bounded and $K(\alpha) \in \exp_c U$ be some element. Then there exist a uniform cover $\beta \in V$ and a uniform cover $\gamma \in U$ cardinality $\leq \tau$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$. Obviously $(\exp_c f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. From $K(\gamma) = \{\langle\gamma'\rangle : \gamma' \subset \gamma\}$ is finite, where $\langle\gamma'\rangle = \{K \in \exp_c X : K \subset \cup\gamma', K \cap G \neq \emptyset, G \in \gamma'\}$ it follows that $K(\gamma)$ is a uniform cover cardinality $\leq \tau$. Thus, $\exp_c f$ is τ-bounded.

2) ⇒ 1). Let $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$ be a τ-bounded. Then, from Remark 0.6.8. ([2]), it follows that (X, U) is a closed subspace of the uniform space $(\exp_c X, \exp_c U)$. It is easy to see that the restriction of the map $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$ to (X, U) is τ-bounded. □

Corollary 2.1. [19] For a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) the following are equivalent:

1. $f : (X, U) \to (Y, V)$ is pre-Lindelöf;
2. $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$ is pre-Lindelöf.

Corollary 2.2. [19] For a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) the following are equivalent:

1. $f : (X, U) \to (Y, V)$ is precompact;
2. $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$ is precompact.

Corollary 2.3. For a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) the following are equivalent:

1. $f : (X, U) \to (Y, V)$ is uniformly perfect;
2. $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$ is uniformly perfect.

Proposition 2.7. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping between uniform spaces (X, U) and (Y, V). If $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ is its exponential mapping, $\exp_c Y = \{K\}$ and $\exp_c(f)$ is τ-bounded, then $(\exp_c X, \exp_c U)$ is also τ-bounded.

Proof. Let $K(\alpha) \in \exp_c U$ be an arbitrary covering. There exists $K(\beta) \in \exp_c V$ and a uniform covering $K(\gamma) \in \exp_c U$ cardinality $\leq \tau$ such that $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. Since $\exp_c Y = \{K\}$, then $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) = K(\gamma) \succ K(\alpha)$, where $K(\gamma) = \{\langle\gamma_i'\rangle : i \in N\}$. Therefore, $(\exp_c X, \exp_c U)$ is τ-bounded. □

Corollary 2.4. [19] Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping between spaces (X, U) and (Y, V). If $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ is its exponential mapping, $\exp_c Y = \{K\}$ and $\exp_c(f)$ is pre-Lindelöf, then $(\exp_c X, \exp_c U)$ is also pre-Lindelöf.

Corollary 2.5. [19] Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping between spaces (X, U) and (Y, V). If $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ is its exponential mapping, $\exp_c Y = \{K\}$ and $\exp_c(f)$ is precompact, then $(\exp_c X, \exp_c U)$ is also precompact.

Corollary 2.6. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping between spaces (X, U) and (Y, V). If $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ is its exponential mapping, $\exp_c Y = \{K\}$ and $\exp_c(f)$ is uniformly perfect, then $(\exp_c X, \exp_c U)$ is compact.

Theorem 2.11. *Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V) and let $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ be its exponential mapping. If $\exp_c f$ and $(\exp_c Y, \exp_c V)$ are τ-bounded, then $(\exp_c X, \exp_c U)$ is also τ-bounded. Conversely, if $(\exp_c X, \exp_c U)$ is τ-bounded, then $\exp_c(f)$ is also τ-bounded.*

Proof. Let $K(\alpha) \in \exp_c U$ be an arbitrary covering. There exists $K(\beta) \in \exp_c V$ and a uniform covering $K(\gamma) \in \exp_c U$ cardinality $\leq \tau$ such that $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. From the covering $K(\beta) \in \exp_c V$ by virtue of the τ-boundedness of $(\exp_c Y, \exp_c V)$, we have a uniform subcovering $K(\beta_0) \in \exp_c V$ cardinality $\leq \tau$. Clearly, $\exp_c(f)^{-1}K(\beta) \in \exp_c U$, so $\exp_c(f)^{-1}K(\beta_0) \wedge K(\gamma) \succ \exp_c(f)^{-1}K(\beta) \wedge K(\gamma)$ and $\exp_c(f)^{-1}K(\beta_0)$ form a covering cardinality $\leq \tau$, since $\exp_c(f)^{-1}(K(\beta)) = \{\langle f^{-1}\beta_1'\rangle, \langle f^{-1}\beta_2'\rangle, \ldots, \langle f^{-1}\beta_n'\rangle\}$. Furthermore, $\exp_c(f)^{-1}(K(\beta)) = \{\langle f^{-1}\beta_1'\rangle, \langle f^{-1}\beta_2'\rangle, \ldots, \langle f^{-1}\beta_n'\rangle\}$ and $K(\gamma) = \{\langle\gamma_1'\rangle, \langle\gamma_2'\rangle, \ldots, \langle\gamma_n'\rangle\}$ are coverings cardinality $\leq \tau$, $(\exp_c f)^{-1}(K(\beta_0)) \wedge K(\gamma)$ is a uniform covering cardinality $\leq \tau$, as the intersection of two uniform coverings cardinality $\leq \tau$. Now, let $K(\alpha) \in \exp_c U$ be some covering. Then, by the τ-bounded property of $(\exp_c X, \exp_c U)$ there exists a covering $K(\gamma) \in \exp_c U$ cardinality $\leq \tau$ such that $K(\gamma) \succ K(\alpha)$. Then, for any uniform covering $K(\beta) \in \exp_c V$, we have $\exp_c(f)^{-1}K(\beta) \wedge K(\gamma) \succ K(\alpha)$. Thus, $\exp_c f$ is a τ-bounded. □

Corollary 2.7. [19] Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V) and let $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ be its exponential mapping. If $\exp_c f$ and $(\exp_c Y, \exp_c V)$ are pre-Lindelöf, then $(\exp_c X, \exp_c U)$ is also pre-Lindelöf. Conversely, if $(\exp_c X, \exp_c U)$ is pre-Lindelöf, then $\exp_c(f)$ is also pre-Lindelöf.

Corollary 2.8. [19] Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V) and let $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ be its exponential mapping. If $\exp_c f$ and $(\exp_c Y, \exp_c V)$ are precompact, then $(\exp_c X, \exp_c U)$ is also precompact. Conversely, if $(\exp_c X, \exp_c U)$ is precompact, then $\exp_c(f)$ is also precompact.

Corollary 2.9. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V) and let $\exp_c f : (\exp_c X, \exp_c U) \to (\exp_c Y, \exp_c V)$, $\exp_c(f)(K) = fK$ be its exponential mapping. If $\exp_c f$ is uniformly perfect and $(\exp_c Y, \exp_c V)$ is compact, then $(\exp_c X, \exp_c U)$ is compact. Conversely, if $(\exp_c X, \exp_c U)$ is compact, then $\exp_c(f)$ is uniformly perfect.

3. Conflict of interests

The author declare that they have no conflict of interest.

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Network-Type and Measure-Functor Properties of the Space of Solutions of Nonlinear Volterra–Fredholm Integro-Differential Equations

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Abstract

We study topological and network-type properties of the solution space of a nonlinear Volterra–Fredholm integro-differential equation on a compact interval. The equation contains both a Volterra memory term and a Fredholm nonlocal term and is considered under continuity assumptions on all nonlinearities and kernels. The space of classical solutions is endowed with the topology inherited from the Banach space of continuous vector-valued functions with the uniform norm.

The first main result shows that the solution space is closed in the ambient Banach space. Consequently, it is a Polish space. A Polish space is a separable, completely metrizable topological space. We then prove a stronger structural fact: the solution space is $\sigma-$compact. Indeed, its intersection with every uniform norm ball is compact; this follows from uniform bounds for the integral terms, a uniform derivative estimate, and the Arzelà–Ascoli theorem. These compactness properties remain valid in the genuinely nonlocal Volterra–Fredholm setting despite the presence of memory and global feedback.

As consequences of the Polish and second countable structure, the solution space has a countable network, a countable $cs-$network, a countable cs^*-network, a countable $cn-$network, and a countable $ck-$network. We further investigate measure-functor constructions. The space of all Borel probability measures on the solution space, equipped with the weak topology, is Polish and hence possesses all of the above countable network-type properties. The same is true for its finite-support subspaces. For each compact bounded stratum of the solution space, the spaces of probability measures and idempotent probability measures are compact metrizable spaces; therefore their network, $cs-$network, cs^*-network, $cn-$network, and $ck-$network are countable. Functorial consequences under continuous transformations of solution spaces are also recorded.

The results extend the topological program previously developed for spaces of solutions of ordinary differential equations to a genuinely nonlocal Volterra–Fredholm setting and reveal additional Polish and compactness features produced by the integral structure.

Keywords: integro-differential equation; Volterra operator; Fredholm operator; solution space; Polish space; $\sigma-$compact space; network; $cs-$network; cs^*-network; $cn-$network; $ck-$network; probability measure; idempotent probability measure.
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1. Introduction. Problem statement

The study of spaces of solutions of differential equations from the viewpoint of general topology has developed into a useful meeting point of differential equations, functional analysis, and functorial topology [1]–[13] of considering only existence, uniqueness, or asymptotic properties of an individual solution, one regards the whole family of solutions as a topological subspace of a suitable function space and investigates its cardinal, convergence, compactness, homotopy, and measure-theoretic properties.

For ordinary differential equations with continuous right-hand sides, this approach leads naturally to solution spaces contained in $C(I, \mathbb{R}^m)$, where I is a compact interval [10], [13]. The work [13] has considered compactness, density, weight, network weight, tightness, connectedness, contractibility, density-type invariants, and several network-type properties of such solution spaces. In particular, the existence of countable networks, $cs-$networks, cs^*-networks, $cn-$networks, and $ck-$networks follows from the metrizable and separable nature of the ambient function space.

The purpose of the present paper is to pass from ordinary differential equations to a genuinely nonlocal class of integro-differential equations. We consider the nonlinear Volterra–Fredholm equation

$$x'(t) = F(t, x(t), (\mathcal{V}x)(t), (\mathcal{W}x)(t)), \qquad t \in I = [a, b], \tag{1.1}$$

where

$$(\mathcal{V}x)(t) = \int_a^t K_V(t, s, x(s))\, ds \tag{1.2}$$

is a Volterra memory term and

$$(\mathcal{W}x)(t) = \int_a^b K_F(t, s, x(s))\, ds \tag{1.3}$$

is a Fredholm nonlocal term. Here

$$F : I \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

$$K_V : D_V \times \mathbb{R}^m \to \mathbb{R}^m, \qquad D_V = \big\{(t, s) \in I^2 : a \le s \le t \le b\big\},$$

and

$$K_F : I^2 \times \mathbb{R}^m \to \mathbb{R}^m$$

are continuous.

This class contains ordinary differential equations as a degenerate case and also includes nonlinear equations with memory and global integral feedback. The integral terms make the equation analytically different from an ODE, but they also produce useful compactness mechanisms. In particular, on a uniformly bounded family of candidate solutions, continuity of the kernels gives uniform bounds for both integral operators, while continuity of F then yields a uniform derivative bound. This allows the Arzelà–Ascoli theorem to be used directly on bounded solution strata.

The first objective of this paper is therefore not merely to repeat the metric argument known for ODE solution spaces, but to identify a stronger topological structure of the solution

set. We prove that the space $\mathcal{S}$ of classical solutions of (1.1) is a closed subspace of $C(I, \mathbb{R}^m)$ and hence is Polish. We then prove that

$$\mathcal{S} = \bigcup_{r=1}^{\infty} \mathcal{S}_r, \qquad \mathcal{S}_r = \{x \in \mathcal{S} : \|x\|_\infty \leq r\},$$

with every $\mathcal{S}_r$ compact. Thus $\mathcal{S}$ is $\sigma-$compact.

These facts immediately strengthen the network-type conclusions. A Polish space is a separable, completely metrizable topological space. Since a Polish space is second countable, a single countable base is simultaneously a countable network, $cs-$network, cs^*-network, $cn-$network, and $ck-$network. This avoids the need to construct a separate family for each network notion.

The second objective concerns spaces of measures over the solution space. For a Polish space X, the space $\mathcal{P}(X)$ of Borel probability measures with the weak topology is itself Polish. We apply this fact to $X = \mathcal{S}$, obtaining all countable network-type properties for the entire probability-measure space, not only for finite-support measures. For the compact solution strata $\mathcal{S}_r$, we also consider the functor $\mathcal{I}$ of idempotent probability measures. Since each $\mathcal{S}_r$ is compact metrizable, $\mathcal{I}(\mathcal{S}_r)$ is compact metrizable and therefore possesses the same countable network-type properties.

The resulting theory thus has two layers. The first is analytic: continuity of the Volterra–Fredholm operators yields closedness and compactness of bounded solution families. The second is topological: Polishness and metrizability automatically transfer strong countability properties to the solution space and to natural measure functors over it.

The paper is organized as follows. Section 2 recalls the network notions and the measure constructions needed below. Section 3 introduces the Volterra–Fredholm equation and proves continuity of the integral operators. Section 4 establishes the closed Polish and $\sigma-$compact structure of the solution space. Section 5 derives the network-type consequences. Section 6 studies probability measures. Section 7 treats idempotent measures on compact solution strata. Section 8 records functorial consequences, and Section 9 contains a discussion of the scope and possible extensions of the results.

2. Preliminaries

Throughout the paper, $I = [a, b] \subset \mathbb{R}$ is a fixed compact interval and $C(I, \mathbb{R}^m)$ is equipped with the uniform norm

$$\|x\|_\infty = \sup_{t \in I} \|x(t)\|.$$

This is a separable Banach space and therefore a Polish space.

Definition 2.1. A family $\mathcal{N}$ of subsets of a topological space X is called a network if for every $x \in X$ and every neighborhood U of x there exists $N \in \mathcal{N}$ such that

$$x \in N \subset U.$$

Definition 2.2. A family $\mathcal{N}$ is a $cs-$network at $x \in X$ if for every sequence (x_j) converging to x and every neighborhood U of x there exists $N \in \mathcal{N}$ such that $N \subset U$ and

$$\{x\} \cup \{x_j : j \geq j_0\} \subset N$$

for some j_0.

Definition 2.3. A family $\mathcal{N}$ is a cs^*-network at x if, whenever $x_j \to x$ and U is a neighborhood of x, there are a subsequence (x_{j_k}) and $N \in \mathcal{N}$ such that

$$\{x_{j_k} : k \in N\} \subset \mathcal{N} \subset U.$$

Definition 2.4. A family $\mathcal{N}$ is a *$cn-$network* at x if, for every neighborhood O_x of x, the set

$$\cup\{N \in \mathcal{N} : x \in N \subset O_x\}$$

is a neighborhood of x.

Definition 2.5. A family $\mathcal{N}$ is a $ck-$network at x if, for every neighborhood O_x of x, there exists a neighborhood $U_x \subset O_x$ such that for each compact $K \subset U_x$ there is a finite subfamily $\mathcal{F} \subset \mathbb{N}$ satisfying

$$x \in \cap\mathcal{F}, \qquad K \subset \cup\mathcal{F} \subset O_x.$$

Every countable base is simultaneously a countable network, a countable $cs-$network, a countable cs^*-network, a countable $cn-$network, and a countable $ck-$network.

For a separable metric space X, let $\mathcal{P}(X)$ denote the space of Borel probability measures on X endowed with the weak topology. If X is Polish, then $\mathcal{P}(X)$ is Polish. For $n \in \mathbb{N}$, we put

$$\mathcal{P}_n(X) = \big\{\mu \in \mathcal{P}(X) : |\operatorname{supp}\mu| \le n\big\}.$$

If X is a compact Hausdorff space, let $\mathcal{I}(X)$ be the space of idempotent probability measures (Maslov measures) on X. A functional $\mu : C(X) \to \mathbb{R}$ is idempotent if

$$\mu(c_X) = c, \qquad \mu(c \odot \varphi) = c \odot \mu(\varphi), \qquad \mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi),$$

where

$$u \oplus v = \max\big\{u, v\big\}, \qquad u \odot v = u + v.$$

The space $\mathcal{I}(X)$ is endowed with the topology of pointwise convergence. For compact metrizable X, the space $\mathcal{I}(X)$ is compact metrizable.

3. The Volterra–Fredholm equation

Let the functions F, K_V, and K_F satisfy the continuity assumptions stated in the introduction. We define the operators $\mathcal{V}$ and $\mathcal{W}$ by (1.2) and (1.3).

A function $x \in C^1(I, \mathbb{R}^m)$ is called a classical solution of (1.1) if the equation holds for every $t \in I$.

Let

$$\mathcal{S} = \big\{x \in C(I, \mathbb{R}^m) : x \in C^1(I, \mathbb{R}^m) \text{ and } x \text{ satisfies (1.1)}\big\}.$$

We equip $\mathcal{S}$ with the subspace topology inherited from $C(I, \mathbb{R}^m)$.

Lemma 3.1. The maps

$$\mathcal{V}, \mathcal{W} : C(I, \mathbb{R}^m) \to C(I, \mathbb{R}^m)$$

are continuous with respect to the uniform norm.

Proof. Let $x_j \to x$ uniformly as $j \to \infty$. Then the sequence (x_j) is uniformly bounded: there exists $R > 0$ such that

$$x_j(I) \cup x(I) \subset \overline{B}_R(0)$$

for all sufficiently large j.

The function K_V is uniformly continuous on the compact set

$$D_V \times \overline{B}_R(0).$$

Hence,

$$\sup_{(t,s)\in D_V} \|K_V(t,s,x_j(s)) - K_V(t,s,x(s))\| \longrightarrow 0 \text{ as } j \to \infty.$$

Therefore,

$$\|\mathcal{V}x_j - \mathcal{V}x\|_\infty \leq (b-a) \sup_{(t,s)\in D_V} \|K_V(t,s,x_j(s)) - K_V(t,s,x(s))\| \to 0 \text{ as } j \to \infty.$$

The proof for $\mathcal{W}$ is identical, using the compact set $I^2 \times \overline{B}_R(0)$. □

Lemma 3.2. Define

$$\mathcal{G}(x)(t) = F(t, x(t), (\mathcal{V}x)(t), (\mathcal{W}x)(t)).$$

Then

$$\mathcal{G} : C(I, \mathbb{R}^m) \to C(I, \mathbb{R}^m)$$

is continuous in the uniform norm.

Proof. Let $x_j \to x$ uniformly as $j \to \infty$. By Lemma 3.1,

$$\mathcal{V}x_j \to \mathcal{V}x, \qquad \mathcal{W}x_j \to \mathcal{W}x$$

uniformly. All values of the four arguments of F therefore remain in a common compact subset of $I \times (\mathbb{R}^m)^3$. Uniform continuity of F on this compact set yields

$$\|\mathcal{G}(x_j) - \mathcal{G}(x)\|_\infty \to 0 \text{ as } j \to \infty.$$

□

4. Polish and compactness properties of the solution space

The differential equation (1.1) is equivalent to the integral identity equation

$$x(t) = x(a) + \int_a^t F(\tau, x(\tau), (\mathcal{V}x)(\tau), (\mathcal{W}x)(\tau))\, d\tau.$$

Theorem 4.1. The solution space $\mathcal{S}$ is closed in $C(I, \mathbb{R}^m)$. Consequently, $\mathcal{S}$ is a Polish space.

Proof. Let $x_j \in \mathcal{S}$ and suppose

$$x_j \to x \text{ as } j \to \infty$$

uniformly in $C(I, \mathbb{R}^m)$. Since every x_j is a solution,

$$x_j(t) = x_j(a) + \int_a^t \mathcal{G}(x_j)(\tau)\, d\tau.$$

By Lemma 3.2,

$$\mathcal{G}(x_j) \to \mathcal{G}(x)$$

uniformly. Passing to the limit gives

$$x(t) = x(a) + \int_a^t \mathcal{G}(x)(\tau)\, d\tau.$$

The integrand is continuous, hence $x \in C^1(I, \mathbb{R}^m)$ and

$$x'(t) = \mathcal{G}(x)(t).$$

Thus $x \in \mathcal{S}$, proving that $\mathcal{S}$ is closed.

The space $C(I, \mathbb{R}^m)$ is Polish. A closed subspace of a Polish space is Polish, so $\mathcal{S}$ is Polish. □

For $r \in \mathcal{N}$ define

$$\mathcal{S}_r = \{x \in \mathcal{S} : \|x\|_\infty \le r\}.$$

Theorem 4.2. For every $r \in \mathcal{N}$, the set $\mathcal{S}_r$ is compact in $C(I, \mathbb{R}^m)$.

Proof. Fix r. Since K_V is continuous on the compact set

$$D_V \times \overline{B}_r(0),$$

there exists $M_{V,r} > 0$ such that

$$\|K_V(t, s, u)\| \le M_{V,r}.$$

Similarly, there exists $M_{F,r} > 0$ such that

$$\|K_F(t, s, u)\| \le M_{F,r}$$

on $I^2 \times \overline{B}_r(0)$.

Hence for every $x \in \mathcal{S}_r$,

$$\|(\mathcal{V}x)(t)\| \le (b-a)M_{V,r},$$

and

$$\|(\mathcal{W}x)(t)\| \le (b-a)M_{F,r}.$$

Therefore, all arguments of F belong to the compact set

$$I \times \overline{B}_r(0) \times \overline{B}_{(b-a)M_{V,r}}(0) \times \overline{B}_{(b-a)M_{F,r}}(0).$$

By continuity of F, there exists $M_r > 0$ such that

$$\|x'(t)\| \le M_r$$

for all $x \in \mathcal{S}_r$ and $t \in I$.

Thus $\mathcal{S}_r$ is uniformly bounded and equi-Lipschitz:

$$\|x(t) - x(s)\| \le M_r|t - s|.$$

By the Arzelà–Ascoli theorem, $\mathcal{S}_r$ is relatively compact in $C(I, \mathbb{R}^m)$. It is closed by Theorem 4.1 and by the closedness of the uniform norm ball. Hence $\mathcal{S}_r$ is compact. □

Corollary 4.1. The solution space $\mathcal{S}$ is $\sigma-$compact:

$$\mathcal{S} = \bigcup_{r=1}^{\infty} \mathcal{S}_r.$$

Remark 4.1. No uniqueness assumption for (1.1) is needed in Theorems 4.1–4.2. The results concern the full family of all classical solutions on the fixed compact interval.

5. Network-type properties

Theorem 5.1. The space $\mathcal{S}$ has

1. a countable network;
2. a countable $cs-$network;
3. a countable cs^*-network;
4. a countable $cn-$network;
5. a countable $ck-$network.

Proof. By Theorem 4.1, $\mathcal{S}$ is Polish and hence second countable. Let $\mathcal{B}$ be a countable base for $\mathcal{S}$.

Every base is a network. If $x_j \to x$ as $j \to \infty$ and U is a neighborhood of x, choose $B \in \mathcal{B}$ with

$$x \in B \subset U.$$

Eventually $x_j \in B$, so $\mathcal{B}$ is a $cs-$network. The same argument immediately gives the cs^*-property.

For the $cn-$property, the union of all $B \in \mathcal{B}$ satisfying $x \in B \subset U$ is an open neighborhood of x contained in U.

Finally, every base is a $ck-$network. If $\mathcal{B}$ is a base and O_x is a neighborhood of x, we choose $x \in B \subset O_x$, $B \in \mathcal{B}$, we set $U_x = B$, and for any compact $K \subset U_x$ we take $\mathcal{F} = \{B\}$. Then automatically, we have $x \in B$, $K \subset B \subset O_x$. □

Corollary 5.1. For every $r \in \mathcal{N}$, the compact solution stratum $\mathcal{S}_r$ has all five countable network-type properties listed in Theorem 5.1.

6. Probability measures on the solution space

Let $\mathcal{P}(\mathcal{S})$ denote the space of all Borel probability measures on $\mathcal{S}$ with the weak topology.

Theorem 6.1. *The space $\mathcal{P}(\mathcal{S})$ is Polish. Consequently, it has a countable network, a countable $cs-$network, a countable cs^*-network, a countable $cn-$network, and a countable $ck-$network.*

Proof. By Theorem 4.1, $\mathcal{S}$ is Polish. A standard theorem on probability measures on Polish spaces states that $\mathcal{P}(\mathcal{S})$, endowed with the weak topology, is again Polish. Hence it is second countable. The conclusion follows exactly as in Theorem 5.1. □

Corollary 6.1. For every $n \in \mathcal{N}$, the finite-support subspace

$$\mathcal{P}_n(\mathcal{S}) = \{\mu \in \mathcal{P}(\mathcal{S}) : |\operatorname{supp}\mu| \leq n\}$$

has all five countable network-type properties.

Proof. Every subspace of a second countable space is second countable. □

Corollary 6.2. For every $r \in \mathcal{N}$, the space $\mathcal{P}(\mathcal{S}_r)$ is compact metrizable and has all five countable network-type properties.

7. Idempotent probability measures on compact solution strata

The full idempotent probability-measure functor is naturally formulated on compact Hausdorff spaces. The compact sets $\mathcal{S}_r \neq \emptyset$ obtained in Theorem 4.2 therefore provide canonical domains for this construction.

Theorem 7.1. For every $r \in \mathcal{N}$ such that $\mathcal{S}_r \neq \emptyset$, the space $\mathcal{I}(\mathcal{S}_r)$ is compact metrizable.

Proof. By Theorem 4.2, $\mathcal{S}_r$ is compact metrizable. The metrization theorem for idempotent probability measures on compact metric spaces implies that $\mathcal{I}(\mathcal{S}_r)$ is metrizable in its pointwise convergence topology. The idempotent probability-measure functor maps compacta to compacta, so $\mathcal{I}(\mathcal{S}_r)$ is compact. □

Corollary 7.1. For every $r, n \in \mathcal{N}$, both $\mathcal{I}(\mathcal{S}_r)$ and its finite-support subspace

$$\mathcal{I}_n(\mathcal{S}_r) = \{\mu \in \mathcal{I}(\mathcal{S}_r) : |\operatorname{supp}\mu| \leq n\}$$

have a countable network, a countable $cs-$network, a countable cs^*-network, a countable $cn-$network, and a countable $ck-$network.

8. Functorial consequences

Let $\mathcal{S}$ and $\mathcal{T}$ be solution spaces of two Volterra–Fredholm integro-differential equations of the form (1.1). Suppose the map

$$h : \mathcal{S} \to \mathcal{T}$$

is continuous. Then the push-forward map

$$\mathcal{P}(h) : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{T}), \qquad \mathcal{P}(h)(\mu)(A) = \mu(h^{-1}(A)),$$

is continuous in the weak topology.

If, in addition, h maps a compact solution stratum $\mathcal{S}_r$ into a compact solution stratum $\mathcal{T}_q$, then the induced map

$$\mathcal{I}(h) : \mathcal{I}(\mathcal{S}_r) \to \mathcal{I}(\mathcal{T}_q), \qquad \mathcal{I}(h)(\mu)(\varphi) = \mu(\varphi \circ h)$$

is continuous.

Proposition 8.1. The probability-measure construction preserves the Polish character of the solution spaces considered above, while both the probability-measure and idempotent probability-measure constructions preserve compact metrizability on compact solution strata.

Proof. The first assertion follows from the general theorem that the probability space over a Polish space is Polish. The second follows from the compact-metrizable character of $\mathcal{S}_r$ and the standard functorial properties of $\mathcal{P}$ and $\mathcal{I}$ on compact metrizable spaces. □

9. Discussion

The passage from ordinary differential equations to Volterra–Fredholm integro-differential equations changes the analytic content of the solution-space problem in an essential way. For an ODE, the basic network conclusions may be read directly from the fact that the solution set is a subspace of a separable metric function space. The same observation remains true for (1.1), but by itself it would produce only a formal extension.

The stronger point is that the integral equation permits a direct proof that the solution space is closed. Uniform convergence of solutions implies uniform convergence of the Volterra and Fredholm terms, because the kernels are uniformly continuous on compact subsets. Passing to the limit in the integral form of the equation then shows that the uniform limit is again a classical solution. This yields completeness of the solution space in a compatible metric and hence Polishness.

A second structural gain is the compactness of the bounded strata $\mathcal{S}_r$. The boundedness of the solutions forces the kernel values to remain in compact sets, which bounds both integral terms. The right-hand side of the differential equation is then uniformly bounded, yielding a uniform Lipschitz constant for the whole stratum. Arzelà–Ascoli therefore converts a simple a priori bound into compactness. As a consequence, the solution space is not merely separable and metrizable but $\sigma-$compact Polish.

This distinction is important for the measure constructions. For the full solution space, Polishness gives a well-behaved space of all Borel probability measures, a result stronger than statements restricted to finite-support measures. On the other hand, idempotent probability measures are most naturally treated over compacta. The compact solution strata provide exactly the appropriate objects for the idempotent-measure functor, without imposing an artificial global compactness assumption on the whole solution space.

The network-type conclusions themselves then become transparent. Once a space is second countable, a countable base simultaneously provides all five network structures considered in this paper. This also avoids technical difficulties that can arise when one tries to prove the $ck-$network property from a family of metric balls centered at a dense set. The compact-cover condition in the definition of a $ck-$network is most cleanly handled by working with a base.

The continuity assumptions imposed here are deliberately minimal for the topological statements. No global Lipschitz condition and no uniqueness theorem are required. If additional Lipschitz hypotheses are imposed, one may obtain continuous dependence on initial data and parameters, and then study the induced maps between the corresponding solution spaces. Such continuous solution operators would interact naturally with the probability and idempotent probability functors considered above.

Several extensions are possible. The same proof scheme applies to systems with several Volterra and Fredholm terms, vector-valued kernels, and parameter-dependent equations, provided the nonlinear operators are continuous in the uniform topology. It is also natural to ask whether analogous results remain valid for delay integro-differential equations, fractional Volterra equations, equations in Banach spaces, or solution spaces endowed with compact-open rather than uniform topologies. In infinite-dimensional state spaces, compactness of bounded solution strata is no longer automatic and would require compactness or condensing assumptions on the integral operators.

10. Conclusion

We have developed a topological theory for the space of classical solutions of a nonlinear Volterra–Fredholm integro-differential equation on a compact interval. The equation combines local differential dynamics with a Volterra memory term and a Fredholm nonlocal term.

The solution space is shown to be closed in the separable Banach space $C(I, \mathbb{R}^m)$ and hence Polish. More strongly, every uniformly bounded solution stratum is compact, which implies that the full solution space is $\sigma-$compact. These structural results provide the foundation for all subsequent network and measure-theoretic conclusions.

As a consequence of second countability, the solution space has a countable network, $cs-$network, cs^*-network, $cn-$network, and $ck-$network. The space of all Borel probability measures on the solution space is again Polish and has the same countable network-type properties; the conclusion therefore goes beyond finite-support probability measures.

For each compact bounded solution stratum, both the ordinary probability-measure space and the idempotent probability-measure space are compact metrizable. Their network-type invariants are consequently countable as well. Continuous transformations between solution spaces induce the usual continuous push-forward maps on these measure spaces, giving a natural functorial interpretation of the results.

Thus, the network-type program for ODE solution spaces extends to a nonlocal integro-differential setting, but the Volterra–Fredholm structure also yields additional Polish, compactness, and measure-functor properties. These results suggest further study of delay, fractional, parameter-dependent, and infinite-dimensional integro-differential equations from the same topological viewpoint.

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Basic Properties of the Normalized Ricci Flow on Reductive Homogeneous Spaces

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Abstract

In this paper we present tools for reducing the normalized Ricci flow equation to a system of nonlinear autonomous differential equations on reductive homogeneous spaces. Methods for equivalent transforming them into systems with polynomial right-hand sides are also discussed. In general case we also consider the interpretation of the geometric concepts from the perspective of the theory of dynamical systems, specifically, the one-to-one correspondence between the invariant Riemannian metrics of a given reductive homogeneous space and the singular (equilibrium) points of the resulting dynamical system, and the property of the metric volume function serving as a first integral of that dynamical system.

Keywords: Ricci flow, Homogeneous space, Einstein metric, Dynamical system, First integral, Singular point.
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1. Introduction

The normalized Ricci flow equation

$$\frac{\partial}{\partial t}\,\mathbf{g}(t) = -2\,\mathrm{Ric}_{\mathbf{g}} + 2\,\mathbf{g}(t)\frac{S_{\mathbf{g}}}{n}, \tag{1.1}$$

introduced by Hamilton in [6] in 1982, where $\mathbf{g}(t)$ is one-parameter family of Riemannian metrics on a n-dimensional Riemannian manifold M, $\mathrm{Ric}_{\mathbf{g}}$ and $S_{\mathbf{g}}$ are respectively the Ricci tensor and the scalar curvature of the metric $\mathbf{g}$, remains a powerful research tool in modern Riemannian geometry. A recent survey on Ricci flows can be found in [4].

In [3] the idea of reducing the normalized Ricci flow to a system of three nonlinear autonomous differential equations for the specific case of so-called generalized Wallach spaces was first proposed and realized. Since then, this system has been studied from various perspectives in our subsequent papers (the bibliography is in [2]) and by other authors. However certain basic facts and technical details were omitted from the text as being non-essential. In this article, we aim to fill this gap by considering the most general case of reductive homogeneous spaces.

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2. Preliminaries and Definitions

Definition 2.1. A Riemannian metric $\mathbf{g}$ of a Riemannian manifold M is called an *Einstein metric*, if $\mathrm{Ric}_{\mathbf{g}} = \lambda \cdot \mathbf{g}$ for some $\lambda \in \mathbb{R}$.

Assume that G is a Lie group and H is a closed subgroup of G. Let G be equipped with a topology τ_G. Consider the set

$$G/H = \{gH \mid g \in G\}$$

of cosets and the natural projection

$$\pi \colon G \to G/H, \quad \pi \colon g \mapsto gH,$$

being a surjection. It is well-known that G/H becomes a Hausdorff topological space in the topology

$$\tau_{G/H} = \{U \in G/H \mid \pi^{-1}(U) \in \tau_G\}$$

induced by τ_G.

Moreover, π is continuous with respect to $\tau_{G/H}$. Therefore G/H is compact if so is G.

The quotient G/H inherits also a smooth structure from G, and hence turns into a smooth manifold.

Definition 2.2. Let G be equipped with a group operation $\circ$ and the group identity e.

1. A *left action* of G on M is a smooth map $G \times M \to M$, $(g, m) \mapsto g \cdot m$, such that $e \cdot m = m$ and $(g_1 \circ g_2) \cdot m = g_1 \cdot (g_2 \cdot m)$ for all $g_1, g_2 \in G$ and $m \in M$.

2. An action is *transitive* if for any $m_1, m_2 \in M$ there exists $g \in G$ such that $g \cdot m_1 = m_2$.

3. M is called a *homogeneous space*, if M is diffeomorphic to a quotient space G/H, where G acts transitively on M and H is the isotropy subgroup at an arbitrary point $m \in M$, i. e. $H = \{g \in G \mid g \cdot m = m\}$.

Definition 2.3. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lee algebras of the Lie groups G and H. Let $\mathfrak{g}$ be equipped with a Lie bracket $[\cdot, \cdot]$.

1. Any mapping $G \to G$ being a diffeomorphism and a group isomorhism is said to be an *automorphism* of G; for a fixed $x \in G$ an automorphism $I_x \colon G \to G$, $I_x : g \mapsto xgx^{-1}$, is called *inner*. A homomorphism $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$, defined as the differential

   $$\mathrm{Ad}(x) = (dI_x)_e \colon \mathfrak{g} \to \mathfrak{g}$$

   of I_x at e is called an *adjoint representation* of the Lie group G in $\mathfrak{g}$.

2. Another homomorphism $\mathrm{ad}(X) \colon \mathfrak{g} \to \mathfrak{g}$, defined as

   $$\mathrm{ad}(X)(Y) = [X, Y],$$

   where $X, Y \in \mathfrak{g}$, is called an *adjoint representation* of the Lie algebra $\mathfrak{g}$.

3. The *Killing form* of the Lie algebra $\mathfrak{g}$ is a bilinear symmetric function $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, defined as

   $$B(X, Y) = \mathrm{trace}\,(\mathrm{ad}(X) \circ \mathrm{ad}(Y))\,.$$

Definition 2.4. A homogeneous space G/H is said to be *reductive*, if there exists a subspace $\mathfrak{p}$ in $\mathfrak{g}$ such that

1. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ (the direct sum of subspaces);
2. $\mathrm{Ad}(H)\mathfrak{p} \subset \mathfrak{p}$ that means $\mathrm{Ad}(h)(X) \in \mathfrak{p}$ for all $h \in H$ and $X \in \mathfrak{p}$.

Definition 2.5. A subspace $W \subset \mathfrak{g}$ is said to be $\mathrm{Ad}(H)$-*invariant*, if $\mathrm{Ad}(H)W \subset W$. A subspace W is *irreducible* if $\mathrm{Ad}(H)W \subset W$ implies either $W = \{0\}$, either $W = \mathfrak{g}$.

Assume for the sequel that G is compact and semi-simple. Then $\mathfrak{p}$ can be chosen to be orthogonal to $\mathfrak{h}$ with respect to the inner product

$$\langle \cdot, \cdot \rangle = -B(\cdot, \cdot) \geq 0$$

on $\mathfrak{g}$ introduced by using the Killing form B, since $B \leq 0$ due to the compactness of G and $B \neq 0$ due to the semi-simplicity of G.

Suppose that $\mathfrak{p}$ decomposes into a direct sum

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_q$$

of pairwise nonequivalent irreducible $\mathrm{Ad}(H)$-invariant modules $\mathfrak{p}_1, \ldots, \mathfrak{p}_q$ of the dimensions $d_i = \dim \mathfrak{p}_i$ such that

$$d_1 + \cdots + d_q = n = \dim(G/H).$$

Assume that an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle = -B(\cdot, \cdot)$ was chosen on $\mathfrak{g}$. Then any G-invariant Riemannian metric $\mathbf{g}(\cdot, \cdot)$ on G/H is determined by an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{p}$:

$$\mathbf{g}(\cdot, \cdot) = x_1 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_1} + \cdots + x_q \langle \cdot, \cdot \rangle|_{\mathfrak{p}_q}, \tag{2.1}$$

where $x_1, \ldots, x_q$ are positive real numbers. Since any G-invariant symmetric covariant 2-tensor on G/H admits the same form as the Riemannian metric (which is a partial case), for the Ricci tensor $\mathrm{Ric}_{\mathbf{g}}$ we have

$$\mathrm{Ric}_{\mathbf{g}}(\cdot, \cdot) = x_1 \mathbf{r}_1 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_1} + \cdots + x_q \mathbf{r}_q \langle \cdot, \cdot \rangle|_{\mathfrak{p}_q}, \tag{2.2}$$

where $\mathbf{r}_i = \mathbf{r}_i(x_1, \ldots, x_q) \in \mathbb{R}$ is the component of the Ricci tensor corresponding to the module $\mathfrak{p}_i$, $i = 1, \ldots, q$.

For any questions concerning homogeneous spaces and invariant Einstein metrics on them we refer the readers to the best and new monograph by Berestovskiĭ and Nikonorov [5], published in 2020.

Using representations (2.1), (2.2) and the expression

$$S_{\mathbf{g}} = d_1 \mathbf{r}_1 + \cdots + d_q \mathbf{r}_q$$

the normalized Ricci flow (1.1) can be reduced to a system of q nonlinear ordinary differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_q) := -2x_i \left(\mathbf{r}_i - \frac{S_{\mathbf{g}}}{n} \right) \tag{2.3}$$

with respect to the unknowns $x_1, \dots, x_q$. According to [9] the following explicit expressions are known for $\mathbf{r}_i$ and $S_{\mathbf{g}}$:

$$\mathbf{r}_i = \frac{1}{2x_i} + \frac{1}{4d_i} \sum_{j,k} \frac{x_i}{x_j x_k} A^i_{jk} - \frac{1}{2d_i} \sum_{j,k} \frac{x_j}{x_i x_k} A^j_{ik}, \tag{2.4}$$

$$S_{\mathbf{g}} = \frac{1}{2} \sum_m \frac{d_m}{x_m} - \frac{1}{4} \sum_{m,j,k} \frac{x_k}{x_m x_j} A^k_{mj}, \tag{2.5}$$

where the sums are taken over $i, j, k, m \in \{1, \dots, q\}$, positive numbers A^k_{ij} depend on the choice of the vector space bases in $\mathfrak{p}_1, \dots, \mathfrak{p}_q$.

Definition 2.6. The function $V = \prod_{i=1}^q x_i^{d_i}$ is called the volume function of the metric (2.1).

Sometimes we deal with an autonomous differential system of the form

$$\frac{dx_i}{dt} = \mathcal{R}(x_1, \dots, x_q) \cdot f_i(x_1, \dots, x_q), \qquad i = 1, \dots, q, \tag{2.6}$$

where $\mathcal{R}(x_1, \dots, x_q) \neq 0$. In such systems the factor $\mathcal{R}(x_1, \dots, x_q)$ can be cancelled.

Lemma 2.1. The system (2.6) is equivalent to the system

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_q), \qquad i = 1, \dots, q. \tag{2.7}$$

Although the proof of this lemma is well known we reproduce it here for the convenience of the readers.

Proof. Introduce the new variable $\tau = t \cdot \mathcal{R}(x_1, \dots, x_q)$. Then obviously

$$\frac{dx_i}{dt} = \frac{dx_i}{d\tau} \cdot \frac{d\tau}{dt} = \frac{dx_i}{d\tau} \cdot \mathcal{R}(x_1, \dots, x_q).$$

Substitute this into (2.6). Then we obtain

$$\frac{dx_i}{d\tau} \cdot \mathcal{R}(x_1, \dots, x_q) = \mathcal{R}(x_1, \dots, x_q) \cdot f_i(x_1, \dots, x_q),$$

equivalently

$$\frac{dx_i}{d\tau} = f_i(x_1, \dots, x_q).$$

Since the autonomous system does not depend on the time parametrization, we can put $\tau := t$. □

3. Main Results

Theorem 3.1. *Every invariant Einstein metrics given by (2.1) is a singular point of the differential system (2.3) and vice versa.*

Proof. Denote $\mathbf{x} = (x_1, \dots, x_q)$ and recall the right hand sides of the differential system (2.3):

$$f_i(\mathbf{x}) = -2x_i \left(\mathbf{r}_i - n^{-1} S_{\mathbf{g}} \right), \qquad i = 1, \dots, q.$$

Assume that $(x_1, \dots, x_q)$ are the parameters of an Einstein metric on a given homogeneous space. Then the condition $\mathrm{Ric}_{\mathbf{g}} = \lambda \cdot \mathbf{g}$ to be Einstein implies $\mathbf{r}_1 = \cdots = \mathbf{r}_q = \lambda$ according to (2.1) and (2.2). It follows then

$$S_{\mathbf{g}} = d_1\mathbf{r}_1 + \cdots + d_q\mathbf{r}_q = (d_1 + \cdots + d_q)\lambda = n\lambda,$$

and consequently

$$\mathbf{r}_i - n^{-1}S_{\mathbf{g}} = \mathbf{r}_i - n^{-1}n\lambda = \lambda - \lambda = 0,$$

meaning that $f_i(\mathbf{x}) = 0$ for each $i \in \{1, \dots, q\}$. By definition $\mathbf{x}$ is a singular (equilibrium) point of the differential system (2.3).

Conversely, assume that $\mathbf{x}$ is a singular point of (2.3), i.e. $f_i(\mathbf{x}) = 0$. Then $\mathbf{r}_i - n^{-1}S_{\mathbf{g}} = 0$ for all $i \in \{1, \dots, q\}$, implying the Einstein property

$$\mathbf{r}_1 = \cdots = \mathbf{r}_q = n^{-1}S_{\mathbf{g}}.$$

□

Theorem 3.2. *The function* $V = \prod_{i=1}^{q} x_i^{d_i} = c$ *is a first integral of the differential system* (2.3) *for any constant* $c > 0$.

Proof. Denote $\Phi = V - c$. By definition Φ is a first integral of the system $\frac{dx_i}{dt} = f_i$ if

$$\frac{\partial \Phi}{\partial x_1} f_1 + \cdots + \frac{\partial \Phi}{\partial x_q} f_q \equiv 0.$$

Observe that

$$\frac{\partial \Phi}{\partial x_i} = \frac{\partial V}{\partial x_i} = d_i\, x_1^{d_1} x_2^{d_2} \times \cdots \times x_i^{d_i - 1} \times \cdots \times x_q^{d_q} = \frac{d_i}{x_i}\, V.$$

Substitute $f_i = -2x_i\,(\mathbf{r}_i - n^{-1}S_{\mathbf{g}})$ into $\dfrac{\partial \Phi}{\partial x_i}\, f_i$:

$$\frac{\partial \Phi}{\partial x_i}\, f_i = \frac{d_i}{x_i}\, f_i\, V = -2d_i\, \left(\mathbf{r}_i - n^{-1}S_{\mathbf{g}}\right) V.$$

Since

$$\sum_{i=1}^{q} d_i \left(\mathbf{r}_i - n^{-1}S_{\mathbf{g}}\right) = \sum_{i=1}^{q} d_i\mathbf{r}_i - n^{-1}S_{\mathbf{g}} \sum_{i=1}^{q} d_i = S_{\mathbf{g}} - n^{-1}S_{\mathbf{g}}\, n = 0,$$

we actually get

$$\sum_{i=1}^{q} \frac{\partial \Phi}{\partial x_i} f_i = -2V \sum_{i=1}^{q} d_i \left(\mathbf{r}_i - n^{-1}S_{\mathbf{g}}\right) = -2V \cdot 0 = 0.$$

□

Remark 3.1. The case $c = 1$ in $V = \prod_{i=1}^{q} x_i^{d_i} = c$ is important from the geometrical point of view. This is the case of so-called unit volume metrics which are preserved under the normalized Ricci flow (1.1). Another reason to restrict ourselves to the case $c = 1$ has an explanation: the system with arbitrarily $c \neq 1$ can equivalently be reduced to $c = 1$ using change of variables. The idea was observed in [1] for the case $q = 3$.

Theorem 3.3. The case of invariant Riemannian metrics with $V = c > 0$ can equivalently be reduced to the case $V = 1$ of the unit volume invariant Riemannian metrics.

Proof. By assumption $x_1^{d_1} \times \cdots \times x_q^{d_q} = c$. Using (2.4) and (2.5) we find

$$f_i(x_1, \ldots, x_q) = -1 - \frac{1}{2d_i} \sum_{j,k} \frac{x_i^2}{x_j x_k} A_{jk}^i + \frac{1}{d_i} \sum_{j,k} \frac{x_j}{x_k} A_{ik}^j - \frac{1}{n} \sum_m \frac{x_i d_m}{x_m} + \frac{1}{2n} \sum_{m,j,k} \frac{x_i x_k}{x_m x_j} A_{mj}^k. \quad (3.1)$$

Observe that $f_i(\beta x_1, \ldots, \beta x_q) = f_i(x_1, \ldots, x_q)$ for every i and any real $\beta \neq 0$. Indeed

$$f_i(\beta x_1, \ldots, \beta x_q) = -1 - \frac{1}{2d_i} \sum_{j,k} \frac{(\beta x_i)^2}{\beta x_j \cdot \beta x_k} A_{jk}^i + \frac{1}{d_i} \sum_{j,k} \frac{\beta x_j}{\beta x_k} A_{ik}^j - \frac{1}{n} \sum_m \frac{\beta x_i d_m}{\beta x_m} + \frac{1}{2n} \sum_{m,j,k} \frac{\beta x_i \cdot \beta x_k}{\beta x_m \cdot \beta x_j} A_{mj}^k = f_i(x_1, \ldots, x_q). \quad (3.2)$$

Introduce the new variables $X_i = x_i \cdot \mathcal{R}$ with $\mathcal{R} \neq 0$, where the value of $\mathcal{R}$ will be actualized later. Substitute $x_i = \mathcal{R}^{-1} \cdot X_i$ into the system (2.3) and use the homogeneity property (3.2). Then

$$\frac{dx_i}{dt} = \mathcal{R}^{-1} \cdot \frac{dX_i}{dt} = f_i\left(\mathcal{R}^{-1} \cdot X_1, \ldots, \mathcal{R}^{-1} \cdot X_q\right) = f_i(X_1, \ldots, X_q).$$

This implies

$$\frac{dX_i}{dt} = \mathcal{R} \cdot f_i(X_1, \ldots, X_q), \qquad i = 1, \ldots, q. \quad (3.3)$$

According to Lemma 2.1 introducing the new time $\tau = t \cdot \mathcal{R}$ and substituting the expression

$$\frac{dX_i}{dt} = \frac{dX_i}{d\tau} \cdot \mathcal{R}$$

into the system (3.3) we obtain a new system

$$\frac{dX_i}{d\tau} = f_i(X_1, \ldots, X_q), \qquad i = 1, \ldots, q. \quad (3.4)$$

In the variables $X_1, \ldots, X_q$ the volume is equal to

$$X_1^{d_1} \times \cdots \times X_q^{d_q} = (\mathcal{R}x_1)^{d_1} \times \cdots \times (\mathcal{R}x_q)^{d_q} = \mathcal{R}^n \cdot \left(x_1^{d_1} \times \cdots \times x_q^{d_q}\right) = \mathcal{R}^n c.$$

Require $X_1^{d_1} \times \cdots \times X_q^{d_q} = 1$. Then the actual value of $\mathcal{R}$ can be found from the condition $\mathcal{R}^n c = 1$:

$$\mathcal{R} = \sqrt[n]{c^{-1}}.$$

In the final, we conclude that if introduce the new variables

$$X_i = x_i \cdot c^{-1/n}, \qquad \tau = t \cdot c^{-1/n}$$

then the original differential system (2.3) with $x_1^{d_1} \times \cdots \times x_q^{d_q} = c$ can equivalently be replaced by the differential system (3.4) of the same form, but with the unit volume condition $X_1^{d_1} \times \cdots \times X_q^{d_q} = 1$. Thus we can put $X_i := x_i$ and $\tau := t$ in the differential system (3.4) without loss of generality. □

Remark 3.2. Using the first integral $V = x_1^{d_1} \times \cdots \times x_q^{d_q} = 1$ we can eliminate one of the variables from the original differential system (2.3). Without loss of generality, let us choose x_q e.g.:

$$x_q = \varphi(x_1, \ldots, x_{q-1}) := \left(x_1^{-d_1} \times \cdots \times x_{q-1}^{-d_{q-1}} \right)^{\frac{1}{d_q}}.$$

This allows us to consider a lower dimensional system with respect to the variables $x_1, \ldots, x_{q-1}$:

$$\frac{dx_i}{dt} = \widetilde{f}_i(x_1, \ldots, x_{q-1}), \tag{3.5}$$

where

$$\widetilde{f}_i(x_1, \ldots, x_{q-1}) = f_i\big(x_1, \ldots, x_{q-1}, \varphi(x_1, \ldots, x_{q-1})\big), \qquad i = 1, \ldots, (q-1).$$

Remark 3.3. The one-to-one correspondence between singular points of the differential systems (2.3) and (3.5) can be established as follows. According to Theorem 3.1 singular points are also invariant Riemannian metrics. Invariance implies that the parameters of the metric (2.1) has the form

$$(x_1, \ldots, x_q) = (\gamma_1 p, \ldots, \gamma_q p),$$

where γ_i and p are all positive. Suppose that $(\gamma_1 p, \ldots, \gamma_q p)$ is an isolated singular point of the differential system (2.3) for every fixed $p > 0$. Then a condition arises on the parameter p under which $(\gamma_1 p, \ldots, \gamma_{q-1} p)$ is also an isolated singular point for the differential system (3.5) of lower order. The actual value of p is defined from the condition $V = 1$:

$$(\gamma_1 p)^{d_1} \times \cdots \times (\gamma_q p)^{d_q} = 1.$$

This equation admits a unique root

$$p_0 = \sqrt[n]{\gamma_1^{-d_1} \times \cdots \times \gamma_q^{-d_q}}.$$

Thus $(\gamma_1 p, \ldots, \gamma_{q-1} p)$ is an isolated singular point of (3.5) if and only if $(\gamma_1 p, \ldots, \gamma_q p)$ is an isolated singular point of (2.3) corresponding to the unique value $p = p_0$.

In general, the differential systems (2.3) and (3.5) are extremely complex to study further, featuring nonlinear right-hand sides f_i of a highly complex rational form shown in (3.1) and (3.5). The functions $\widetilde{f}_i(x_1, \ldots, x_{q-1})$ in (3.5) contains also rational powers $x_i^{-\frac{d_i}{d_q}}$ of the variables.

Lemma 2.1 is useful in cases when the functions f_i admit a common factor similar to $\mathcal{R}(x_1, \ldots, x_q)$ providing an opportunity to transform the system (2.3) into a differential system with polynomial right-hand sides.

We demonstrate the realization of this idea for the case of generalized Wallach spaces, for which $q = 3$. Generalized Wallach spaces can be described by three real parameters $a_i = A/d_i \in (0, 1/2)$, $i = 1, 2, 3$, where A is some positive constant (see [8] for definitions and details). It should be noted that in [8] the classification of generalized Wallach spaces was obtained.

In [7, Lemma 2] the expressions

$$\begin{aligned}
\mathbf{r}_1 &= \frac{1}{2x_1} + \frac{a_1}{2}\left(\frac{x_1}{x_2 x_3} - \frac{x_2}{x_3 x_1} - \frac{x_3}{x_1 x_2} \right), \\
\mathbf{r}_2 &= \frac{1}{2x_2} + \frac{a_2}{2}\left(\frac{x_2}{x_3 x_1} - \frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} \right), \\
\mathbf{r}_3 &= \frac{1}{2x_3} + \frac{a_3}{2}\left(\frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_3 x_1} \right)
\end{aligned} \tag{3.6}$$

were found for the components of the Ricci tensor of invariant Riemannian metrics on generalized Wallach spaces. According to (3.6) the functions f_i in the system

$$\frac{dx_i}{dt} = f_i = -2x_i\left(\mathbf{r}_i - \frac{S_{\mathbf{g}}}{n}\right) = -2x_i\left(\mathbf{r}_i - \frac{\mathbf{r}_1 a_1^{-1} + \mathbf{r}_2 a_2^{-1} + \mathbf{r}_3 a_3^{-1}}{a_1^{-1} + a_2^{-1} + a_3^{-1}}\right) \tag{3.7}$$

take the form

$$\begin{aligned} f_i(x_1,x_2,x_3) = -1 - a_i x_i &\left(\frac{x_i}{x_j x_k} - \frac{x_j}{x_k x_i} - \frac{x_k}{x_i x_j}\right) \\ &+ \frac{x_i}{a_1^{-1} + a_2^{-1} + a_3^{-1}}\left[\frac{1}{a_1 x_1} + \frac{1}{a_2 x_2} + \frac{1}{a_3 x_3} - \left(\frac{x_1}{x_2 x_3} + \frac{x_2}{x_3 x_1} + \frac{x_3}{x_1 x_2}\right)\right], \end{aligned} \tag{3.8}$$

where (i,j,k) accepts the values $(1,2,3)$, $(2,3,1)$ and $(3,1,2)$. As can be seen here, the system does not lose its complexity even in the case of $q = 3$. The volume function takes the form

$$V = x_1^{\frac{1}{a_1}} x_2^{\frac{1}{a_2}} x_3^{\frac{1}{a_3}}.$$

$V = 1$ is also a first integral of the differential system (3.7) by Theorem 3.2. Therefore, we can reduce (3.7) to the following system of two nonlinear autonomous differential equations eliminating one of the variables x_i, say x_3 (see also Remark 3.2):

$$\frac{dx_1}{dt} = \widetilde{f}_1(x_1,x_2), \qquad \frac{dx_2}{dt} = \widetilde{f}_2(x_1,x_2), \tag{3.9}$$

where

$$\begin{aligned} \widetilde{f}_1(x_1,x_2) &= f_1\big(x_1,x_2,\varphi(x_1,x_2)\big), \\ \widetilde{f}_2(x_1,x_2) &= f_2\big(x_1,x_2,\varphi(x_1,x_2)\big), \\ \varphi(x_1,x_2) &= x_1^{-\frac{a_3}{a_1}} x_2^{-\frac{a_3}{a_2}}. \end{aligned}$$

Demonstrate now how the differential system (3.7) can equivalently be replaced by a differential system with polynomial right-hand sides. Transforming $f_i(x_1,x_2,x_3)$ in (3.8) equivalently we obtain

$$f_i(x_1,x_2,x_3) = \frac{1}{\mathcal{A}\,x_1x_2x_3}\cdot g_i(x_1,x_2,x_3),$$

where each of the functions

$$g_i = a_i x_i\Big\{a_i\,(a_j+a_k)\,\big(x_j^2 + x_k^2 - x_i^2\big) + a_j x_j\,(x_i - x_k) + a_k x_k\,(x_i - x_j) - 2a_j a_k x_i^2\Big\}$$

is a polynomial in the variables x_1, x_2, x_3 and $\mathcal{A} = a_1a_2 + a_1a_3 + a_2a_3 > 0$. Clearly

$$\mathcal{R}(x_1,x_2,x_3) = \frac{1}{\mathcal{A}\,x_1x_2x_3} > 0$$

in the considered case, since $x_1, x_2, x_3 > 0$ by hypothesis. By Lemma 2.1 it suffices to consider the system

$$\frac{dx_i}{dt} = g_i(x_1,x_2,x_3) \tag{3.10}$$

instead of (3.7), where the fractional functions $f_i(x_1,x_2,x_3)$ given in (3.8) are replaced by the polynomials $g_i(x_1,x_2,x_3)$.

Example 3.1. The interesting case $a_1 = a_2 = a_3 = 1/4$ was studied in [3]. The system (3.9) takes the form

$$\frac{dx_1}{dt} = \frac{1}{3}\left(x_1^2x_2 - 2 + \frac{x_1}{x_2}\right) - \frac{x_1}{6}\left(2x_1^2 - x_2^2 - \frac{1}{x_1^2x_2^2}\right),$$
$$\frac{dx_2}{dt} = \frac{1}{3}\left(x_1x_2^2 - 2 + \frac{x_2}{x_1}\right) - \frac{x_2}{6}\left(2x_2^2 - x_1^2 - \frac{1}{x_1^2x_2^2}\right).$$

The equivalent polynomial system is

$$\frac{dx_1}{dt} = 2x_1^2x_2\left(x_1^2x_2^2 + x_1 - 2x_2\right) + x_1\left(1 + x_1^2x_2^4 - 2x_1^4x_2^2\right),$$
$$\frac{dx_2}{dt} = 2x_2^2x_1\left(x_1^2x_2^2 + x_2 - 2x_1\right) + x_2\left(1 + x_1^4x_2^2 - 2x_1^2x_2^4\right).$$

These systems have the same phase portrait.

Remark 3.4. It should be noted that the Ricci tensor components $\mathbf{r}_i$ need not to be symmetric as in the case of the Stiefel space

$$\mathrm{SO}(n)/\mathrm{SO}(n-2), \qquad n \geq 3.$$

The following expressions hold for invariant Riemannian metric as shown in [10]:

$$\mathbf{r}_1 = \frac{1}{2x_1} - \frac{1}{4(n-2)}\left(\frac{x_3}{x_1x_2} + \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\right),$$
$$\mathbf{r}_2 = \frac{1}{2x_2} - \frac{1}{4(n-2)}\left(\frac{x_3}{x_1x_2} + \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3}\right),$$
$$\mathbf{r}_3 = \frac{1}{2x_3} - \frac{1}{4}\left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right).$$

The scalar curvature $S_{\mathbf{g}} = d_1\mathbf{r}_1 + d_2\mathbf{r}_2 + d_3\mathbf{r}_3$ has the form

$$S_{\mathbf{g}} = \frac{1}{2x_3} - \frac{1}{4}\left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2}\right) + \frac{n-2}{2}\left(\frac{1}{x_1} + \frac{1}{x_2}\right).$$

Substituting these expressions for $\mathbf{r}_i$ and $S_{\mathbf{g}}$ into the general system (2.3) the system $\frac{dx_i}{dt} = f_i(x_1, x_2, x_3)$ of ordinary differential equations can be obtained for $\mathrm{SO}(n)/\mathrm{SO}(n-2)$, $n \geq 3$. As shown in [10] the functions f_i are non-symmetric in x_1, x_2, x_3:

$$f_1 = \frac{(n-1)\left(x_2^2 + x_3^2\right) - (3n-5)x_1^2}{2(n-2)(2n-3)x_2x_3} + \frac{x_1x_2 + (n-2)x_3x_1 - (n-1)x_2x_3}{(2n-3)x_2x_3},$$
$$f_2 = \frac{(n-1)\left(x_3^2 + x_1^2\right) - (3n-5)x_2^2}{2(n-2)(2n-3)x_3x_1} + \frac{x_1x_2 + (n-2)x_2x_3 - (n-1)x_3x_1}{(2n-3)x_3x_1},$$
$$f_3 = \frac{(n-2)(x_1-x_2)^2 + (n-2)(x_1+x_2)x_3 - (n-1)x_3^2}{(2n-3)x_1x_2}.$$

4. Conclusion

We demonstrated how the normalized Ricci flow equation can be reduced to a system of differential equations on homogeneous spaces and discussed basic properties of that system, such as its singular points and first integral. We also presented an example of an equivalent transformation of similar systems into a polynomial system.

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About S-bounded Topological Groups

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Abstract

Topological groups and uniform spaces are closely linked through the concept of a uniform structures. Each topological group naturally gives rise to a uniform structure, which makes it possible to extend the concepts of uniform continuity and completion. In this paper S-bounded topological groups are studied using uniform structures, in particular, it is proved that S-boundedness is preserved under continuous homomorphism into the image and that the completion of any S-bonded topological group is a S-bonded topological group.

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1. Introduction

For covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. Let M subset of X. Then $St(\alpha, M) = \cup\{A \in \alpha : A \cap M \neq \varnothing\}$. Let $x \in X$ be an arbitrary point. Then $St(\alpha, x) = \cup\{A \in \alpha : A \ni x\}$. The symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$. The symbol $\alpha \triangleright \beta$ means that the cover α is a star refinement of the cover β, i.e., for any $x \in X$ there exists $B \in \beta$ such that $St(\alpha, x) \subset B$ and, the symbol $\alpha* \succ \beta$ means that the cover α is a strongly star refinement of the cover β, i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $St(\alpha, A) \subset B$.

A topological space X is said to be:

1. Menger if, for each sequence $(\alpha_n : n \in N)$ of open covers X, there is a sequence $(\beta_n : n \in N)$ such that β_n is a finite subset of α_n for each $n \in N$, and $X = \bigcup_{n\in N} \beta_n$ [8], [15]-[19];

2. Hurewicz if, for each sequence $(\alpha_n : n \in N)$ of open covers X, there is a sequence $(\beta_n : n \in N)$ such that β_n is a finite subset of α_n for each $n \in N$, and each $x \in X$ belongs to all but finitely many sets $\bigcup_{n\in N} \beta_n$ [8], [15]-[19];

3. Rothberger if, for each sequence $(\alpha_n : n \in N)$ of open covers X, there is a sequence $(A_n : n \in N)$ such that $A_n \in \alpha_n$ for each $n \in N$, and $X = \bigcup_{n\in N} A_n$ [8], [15]-[19];

4. Scheepers if, for each sequence $(\alpha_n : n \in N)$ of open covers of X, there exist a sequence $(\beta_n : n \in N)$ of finite subfamilies such that $\beta_n \subset \alpha_n$ for each $n \in N$, and for each finite set $F \subset X$ there is n such that $F \subset \bigcup_{B\in\beta_n} B$ [10], [15].

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Let $(G, \cdot, \tau)$ is a topological group. For $A, B \subset G$, $A \cdot B$ stands for $\{a \cdot b : a \in A, b \in B\}$ and $A \cdot b$ stands for $\{a \cdot b : a \in A\}$. Let $B(e)$ is a neighborhood of unity. Each element $W \in B(e)$ determines three covers of the topological group $(G, \cdot, \tau)$: $\alpha^l_W = \{x \cdot W : x \in G\}$, $\alpha^r_W = \{W \cdot x : x \in G\}$ and $\alpha^t_W = \{x \cdot W \cdot y : x, y \in G\}$. These three systems of covers determine three uniformities on the topological group $(G, \cdot, \tau)$ called the left, right and two-sided uniformities: U_l, U_r and U_t.

A topological group G is said to be:

a) Menger-bounded (or M-bounded) if for each sequence $(U_n : n \in N)$ of neighborhoods of the unit, there exist finite sets $F_n \subset G$, $n \in N$, such that $G = \bigcup_{n \in N} F_n \cdot U_n$ [9], [10];

b) Hurewicz-bounded (or H-bounded) if for each sequence $(U_n : n \in N)$ of neighborhoods of the unit, there exist finite sets $F_n \subset G$, $n \in N$, such that for each $g \in G$, $g \in F_n \cdot U_n$ for all but finitely many n [9], [10];

c) Rothberger-bounded (or R-bounded) if for each sequence $(U_n : n \in N)$ of neighborhoods of the unit, there exist elements $a_n \in G$, $n \in N$, such that $G = \bigcup_{n \in N} a_n \cdot U_n$[9], [10];

d) Scheepers-bounded (or S-bounded) if for each sequence $(U_n : n \in N)$ of neighborhoods of the unit, there exist finite sets $F_n \subset G$, $n \in N$, such that for each finite set $F \subset G$, there is n such that $F \subset F_n \cdot U_n$ [10].

A uniformity on a set $X \neq \varnothing$ is a family U of covers of X which satisfies the following axioms:

(U1) If $\alpha \in U$ and $\alpha \succ \beta$, then $\beta \in U$;

(U2) If $\alpha, \beta \in U$, then there exists $\gamma \in U$ such that $\gamma \succ \alpha$ and $\gamma \succ \beta$;

(U3) If $\alpha \in U$, then there exists $\beta \in U$ such that $\beta* \succ \alpha$;

(U4) For any two points $x, y \in X, x \neq y$ there is an $\alpha \in U$ such that $y \notin \alpha(x)$ and $x \notin \alpha(y)$ [1]–[5].

The pair (X, U) is called a uniform space.

Recall that a uniform space (X, U) is called:

(i) precompact if the uniformity U has a base consisting of finite covers [1], [12];

(ii) totally bounded, if each $\alpha \in U$ has a finite set $M \subset X$ such that $\alpha(M) = X$ [1], [12];

(iii) ω-bounded, if each $\alpha \in U$ has a countable set $M \subset X$ such that $\alpha(M) = X$ [1], [6];

(iv) pre-Lindelöf, if the uniformity U has a base consisting of countable cover [1], [6];

(v) has the uniform Menger property, if for each sequence $(\alpha_n : n \in N) \subset U$ there is a sequence $(\beta_n : n \in N)$ such that for each $n \in N$, β_n is a finite subset of α_n and $\bigcup_{n \in N} \beta_n$ is a cover of X [1], [4], [6];

(vi) has the uniform Hurewicz property, if for each sequence $(\alpha_n : n \in N) \subset U$ there is a sequence $(\beta_n : n \in N)$ such that for each $n \in N$, β_n is a finite subset of α_n and for each $x \in X$ we have $x \in \cup\beta_n$ for all but finitely many n [1], [4], [6];

(vii) uniformly Rothberger space or has the uniform Rothberger property if for each sequence $(\alpha_n : n \in N) \subset U$ there is a sequence $\{A_n\}$ such that for each $n \in N$ $A_n \in \alpha_n$ and $\bigcup_{n \in N} A_n = X$ [1], [6], [11]–[14];

(viii) complete if every Cauchy filter in it converges [1], [3];

(ix) $\aleph_0$-complete if every $\aleph_0$-centered Cauchy filter in it converges [1], [11].

A mapping $f : (X, U) \to (Y, V)$ of the uniform space (X, U) to the uniform space (Y, V) is called a uniformly continuous mapping, if every $\beta \in U$ there is $\alpha \in U$ such that $f\alpha \succ \beta$ [1], [20].

A filter F in a uniform space (X, U) is said to be a Cauchy filter, if $\alpha \cap F \neq \varnothing$ for any $\alpha \in U$ [1]; a Cauchy filter F is called a free Cauchy filter if $\cap\{[M] : M \in F\} = \varnothing$, where $[M]$ is the closure of the set M; a point $x \in X$ is said to be an adherent point of a filter F in (X, U), if $x \in \cap\{[M] : M \in F\}$ or equivalently, $x \in X$ is an adherent point of each element $M \in F$ in (X, τ_U) [1]; a point $x \in X$ is said to be a limit of a filter F, if a filter F is finer than the filter F_x of neighborhoods of the point $x \in X$ [3], [5], a filter F is called $\aleph_0$-centered if for any $F' \subset F$ of cardinality $|F'| \leq \aleph_0$, $\cap F' \neq \varnothing$ holds []; a topological group $(G, \cdot, \tau)$ is called Raikov $\aleph_0$-complete if the uniform space (G, U_t) is $\aleph_0$-complete, where U_t is two-sided uniformity on $(G, \cdot, \tau)$ [11].

Let $\widetilde{X}$ be the set of all minimal Cauchy filters in the uniform space (X, U). For each uniform cover $\alpha \in U$ we denote $\widetilde{\alpha} = \{\widetilde{A} : A \in \alpha\}$, where $\widetilde{A} = \{F \in \widetilde{X} : A \in F\}$. The family of covers $\widetilde{B} = \{\widetilde{\alpha} : \alpha \in U\}$ forms a base of some uniformity $\widetilde{U}$ on $\widetilde{X}$ [11].

2. S-bounded groups

Assume that $(G, \cdot, \tau)$ is a topological group. By analogy with the definition of the Scheepers property in topological spaces we introduce the next notion.

A uniform space (G, U_l) is called a uniformly Scheepers space if for each sequence $(\alpha_{U_n} : n \in N)$ of uniform covers of (G, U_l), there exist a sequence $(\beta_{U_n} : n \in N)$ of finite subfamilies such that $\beta_{U_n} \subset \alpha_{U_n}$ for each $n \in N$ and for each finite set $F \subset G$ there is n such that $F \subset \bigcup_{x \cdot U_n \in \beta_{U_n}} x \cdot U_n$.

Proposition 2.1. A topological group G is S-bounded if and only if the uniform space (G, U_l) is uniformly Scheepers, where U_l is the left uniformity.

Proof. Necessity. Let G be an S-bounded topological group and $(\alpha_{U_n} : n \in N) \subset U_l$ be an arbitrary sequence of uniform covers, $\alpha_{U_n} = \{x \cdot U_n : x \in G\}$, U_n is neighborhood of unity. Then there exist finite sets $V_n \subset G$, $n \in N$, such that for each finite set $F \subset G$ there is n such that $F \subset V_n \cdot U_n$. We show that $(\beta_{U_n} : n \in N)$ is the desired sequence of finite subfamilies of the family α_{U_n} for each $n \in N$, $\beta_{U_n} = \{x \cdot U_n : x \in V_n\}$. Indeed, since for each $n \in N$, $F \subset V_n \cdot U_n$, then $F \subset \bigcup_{x \cdot U_n \in \beta_{U_n}} x \cdot U_n$.

Sufficiency. Let (G, U_l) be a uniformly Scheepers space. We show that G is a S-bounded topological group. Let $(U_n : n \in N)$ be an arbitrary sequence of neighborhoods of unity. Then for the sequence of uniform covers $(\alpha_{U_n} : n \in N)$, where $\alpha_{U_n} = \{x \cdot U_n : x \in G\}$ there exist finite subfamilies $(\beta_{U_n} : n \in N)$, $\beta_{U_n} \subset \alpha_{U_n}$ such that for each finite set $F \subset G$ there is $n \in N$, such that $F \subset \bigcup_{x_i \cdot U_n \in \beta_{U_n}} x_i \cdot U_n$, $i = 1, \ldots, m$. For each set $x_i \cdot U_n \in \beta_{U_n}$ we select one element $x_i \in x_i \cdot U_n$ and put $F_{\beta_{U_n}} = \{x_1^{x_1 U_n}, \ldots, x_m^{x_m U_n}\}$. Then for the sequence $(F_{\beta_{U_n}} : n \in N)$ of finite subsets of G it is easy to see that $F \subset F_{\beta_{U_n}} \cdot U_n$, since $F \subset \bigcup_{x_i \cdot U_n \in \beta_{U_n}} x_i \cdot U_n$. □

Proposition 2.2. If (G, τ_{U_l}) is a Scheepers space, then (G, U_l) is uniformly Scheepers.

Proof. Let $(\alpha_{U_n} : n \in N)$ be an arbitrary sequence of uniform covers. Put $\langle \alpha_{U_n} \rangle = \{\langle x \cdot U_n \rangle_{\tau_U} : x \cdot U_n \in \alpha_{U_n}\}$, $\langle x \cdot U_n \rangle_{\tau_{U_l}}$ - interior of the set $x \cdot U_n$. Then $(\langle \alpha_{U_n} \rangle : n \in N)$ is a sequence of open uniform covers of G. Since G is a Scheepers then for covers $(\langle \alpha_{U_n} \rangle : n \in N)$ there is a finite subsets $(\beta_{U_n} : n \in N)$, such that for each finite subsets F there is $n \in N$ such that $F \subset \bigcup_{x \cdot U_n \in \beta_{U_n}} x \cdot U_n$. Thus, (G, U_l) is a uniformly Scheepers. □

Proposition 2.3. If G is precompact, then it is S-bounded.

Proof. Let $(\alpha_{U_n} : n \in N) \subset U_l$ be an arbitrary sequence. Since G is precompact then every cover $\alpha_{U_n} \in U_l$ contains a finite subcover $\alpha^0_{U_n} \in U$. Let $F \subset G$ be an arbitrary finite set. It is easy to see that for each $n \in N$ $F \subset \cup \alpha^0_n$. Thus G is S-bounded. □

Corollary 2.1. If G is compact, then it is S-bounded.

Corollary 2.2. If G is totally bounded, then it is S-bounded.

Proposition 2.4. If G is H-bounded, then G is S-bounded.

Proof. Let $(\alpha_{U_n} : n \in N) \subset U_l$ be an arbitrary sequence of uniform covers. Then there is sequence $(\beta_{U_n} : n \in N)$ such that each β_{U_n} is a finite subset of α_{U_n} and for each $x \in G$ we have $x \in \cup \beta_{U_n}$ for all but finitely many $n \in N$. Let $F = \{x_1, \ldots, x_k\}$ be an arbitrary finite subset. Then for each $x_i \in F$, $i = 1, \ldots, k$ we have $x_i \in \cup \beta_{U_n}$ for all but finitely many $n \in N$. Since for each $i \in \{1, \ldots, k\}$ the number of sets $\cup \beta_{U_n}$ not containing x_i is also finite. Consequently, there exists an index n' such that $F \subset \bigcup_{x \cdot U_{n'} \in \beta_{U_{n'}}} x \cdot U_{n'}$. Thus, G is S-bounded. □

Proposition 2.5. If G is S-bounded, then G is M-bounded.

Proof. Let $(\alpha_{U_n} : n \in N) \subset U_l$ be an arbitrary sequence of covers of G. Then there is a sequence $(\beta_{U_n} : n \in N)$ of finite subfamily such that each $\beta_{U_n} \subset \alpha_{U_n}$ and for each finite set $F \subset G$ there is $n \in N$ such that $F \subset \cup \beta_{U_n}$. We proof that $G = \bigcup_{n \in N} \beta_{U_n}$. Let $x \in G$ be an arbitrary point. Then there is $n \in N$ such that $\{x\} \subset \cup \beta_{U_n}$ i.e. $x \in \bigcup_{n \in N} \beta_{U_n}$. Thus, G is M-bounded. □

Proposition 2.6. Every S-bounded topological group is ω-bounded.

Theorem 2.1. *A topological group G is a S-bounded if and only if the uniform space (G, U_G) is a uniformly Scheepers space, where U_G is the universal uniformity on G.*

Proof. The necessity follows from Proposition 2.2. Conversely, let $(\alpha_{U_n} : n \in N)$ be an arbitrary sequence of open covers. Since the set of all open covers forms the base of the universal uniformity, then $(\alpha_{U_n} : n \in N) \subset U_G$. Therefore, there exist a sequence $(\beta_{U_n} : n \in N)$ of finite subfamilies $\beta_{U_n} \subset \alpha_{U_n}$, $n \in N$, such that for each finite subsets F there is $n \in N$ such that $F \subset \bigcup_{x \cdot U_n \in \beta_{U_n}} x \cdot U_n$. Consequently, G is a S-bounded.

Notice that each S-bounded group is M-bounded but these two classes of spaces are distinct. From the Theorem 2.1. we conclude that the classes of S-bounded and H-bounded topological groups are different. □

Theorem 2.2. *Let G be a topological group. Then for a uniform space (G, U_l) the following conditions are equivalent:*

(a) (G, U_l) is a uniformly Scheepers space;

(b) for each sequence $(\alpha_{U_n} : n \in N) \subset U_l$ *there is a sequence* $(M_n : n \in N)$ *of finite subsets of* (G, U_l) *such that for each finite set* $F \subset G$ *there is* $n \in N$ *such that* $F \subset St(M_n, \alpha_{U_n})$*;*

(c) for each sequence $(\alpha_{U_n} : n \in N) \subset U_l$ *there is a sequence* $(\beta_{U_n} : n \in N)$ *of finite subfamilies* (G, U_l) *such that for each* $n \in N$*,* β_{U_n} *is a finite subfamily* α_{U_n} *and for each finite set* $F \subset G$ *there is* $n \in N$ *such that* $F \subset St(\cup\beta_{U_n}, \alpha_{U_n})$*.*

Proof. $(a) \Rightarrow (b)$. Let $(\alpha_{U_n} : n \in N) \subset U_l$ be a sequence of uniform covers. Apply (a) to the sequence $(\alpha_{U_n} : n \in N) \subset U_l$ of uniform covers, for each $n \in N$ we choose a finite subfamily β_{U_n} of the cover α_{U_n} such that for each finite set $F \subset G$ there is $n \in N$ such that $F \subset \bigcup_{x \cdot U_n \in \beta_{U_n}} x \cdot U_n$. For each $n \in N$ and $x \cdot U_n \in \beta_{U_n}$ we choose an element $x^{\beta_{U_n}}_{x \cdot U_n} \in x \cdot U_n$ and put $M_n = \{x^{\beta_{U_n}}_{x \cdot U_n} : x \cdot U_n \in \beta_{U_n}\}$. Then each M_n is a finite subset of the space (G, U_l). Thus, $F \subset \bigcup_{B \in \beta_n} St(x^{\beta_{U_n}}_{x \cdot U_n}, \alpha_{U_n}) \subset St(M_n, \alpha_{U_n})$.

$(b) \Rightarrow (c)$. Let $(\alpha_{U_n} : n \in N) \subset U_l$ be an arbitrary sequence of uniform covers. Then there is a sequence $(M_n : n \in N)$ of finite subsets of the space (G, U_l) such that for each finite set $F \subset G$ there is $n \in N$ such that $F \subset St(M_n, \alpha_{U_n})$. For each $x' \in M_n$ we choose an element $x \cdot U_n \in \alpha_{U_n}$ such that $x \cdot U_n \ni x'$ and put $\beta_{U_n} = \{x \cdot U_n : x \in M_n\}$. It is clear that β_{U_n} is a finite subfamily of the cover α_{U_n} for each $n \in N$. Thus, for a finite set $F \subset G$ there is $n \in N$ such that $F \subset St(M_n, \alpha_{U_n}) \subset St(\cup\beta_{U_n}, \alpha_{U_n})$.

$(c) \Rightarrow (a)$. Let $(\alpha_{U_n} : n \in N) \subset U_l$ be an arbitrary sequence of uniform covers. Then for each $n \in N$ there is $V_n \in B(e)$ such that $V_n \cdot V_n \subset U_n$. It is easy to see that $\gamma_{V_n} * \succ \alpha_{U_n}$. For the sequence $(\gamma_{V_n} : n \in N) \subset U_l$ there is a sequence $(\lambda_{\gamma_n} : n \in N)$ such that each λ_{V_n} is a finite subfamily of the cover γ_{V_n}, and for each finite set $F \subset G$ there is a $n \in N$ such that $F \subset St(\cup\lambda_{V_n}, \gamma_{V_n})$. Further, for each $n \in N$ and each $x \cdot V_n \in \lambda_{V_n}$ we choose a $x \cdot U_n \in \alpha_{U_n}$ such that $St(x \cdot V_n, \gamma_{V_n}) \subset x \cdot U_n$, and set $\beta_{V_n} = \{x \cdot U_{x \cdot V_n} : x \cdot V_n \in \lambda_{V_n}\}$. It is clear that β_{U_n} is the finite subfamily of the cover α_{U_n} for each $n \in N$. Thus, $F \subset St(\cup\lambda_{V_n}, \gamma_{V_n}) \subset \bigcup_{x \cdot V_n \in \lambda_{V_n}} St(x \cdot V_n, \gamma_{V_n}) \subset \bigcup_{x \cdot V_n \in \lambda_n} x \cdot U_{x \cdot V_n}$. □

Theorem 2.3. *Every subgroup of a topological group* G *is* S*-bounded.*

Theorem 2.4. *Let* $f : G \to G'$ *be a continuous homomorphism of a topological group* G *to a topological group* G'*. If* G *is* S*-bounded topological group, then* G' *is also* S*-bounded topological group.*

Proof. Let $f : G \to G'$ be a continuous homomorphism and $(\beta_{V_n} \mid n \in N) \subset V_l$ be an arbitrary sequence of uniform covers, $f : (G, U_l) \to (G', V_l)$. Then the preimage $f^{-1}\beta_{V_n}$ of each uniform cover β_{V_n} is a uniform cover, i.e. $f^{-1}\beta_{V_n} \in U_l$ for each $n \in N$. Since G is S-bounded, for the sequence $(f^{-1}\beta_{V_n} \mid n \in N) \subset U_l$ of uniform covers there is a sequence $(\alpha_{f^{-1}V_n} \mid n \in N)$ of finite subfamilies such that $\alpha_{f^{-1}V_n} \subset f^{-1}\beta_{V_n}$ for each $n \in N$. For each finite set $M \subset G$ there is $n \in N$ such that $M \subset \bigcup_{x_i \cdot f^{-1}V_n \in \alpha_{f^{-1}V_n}} x_i \cdot f^{-1}V_n$.

Let $F \subset Y$ be an arbitrary finite set. Then there is a finite set $f^{-1}F \subset G$. For each element $y_i \in F$, $F = \{y_1, \ldots, y_k\}$, $i = 1, \ldots, k$, from the set $f^{-1}y_i$, we choose one element x_i and put $M = \{x_1, \ldots, x_k\}$. Since (G, U_l) is uniformly Scheepers, there is $n \in N$ such that $M \subset \bigcup_{x_i \cdot f^{-1}V_n \in \alpha_{f^{-1}V_n}} x_i \cdot f^{-1}V_n$, $i = 1, \ldots, k$. Thus $fM \subset \bigcup_{f(x_i)V_n \in f\alpha_{f^{-1}V_n}} f(x_i)V_n$. Consequently, G' is S-bounded.

□

Theorem 2.5. *The completion of* S*-bounded topological group is* S*-bounded topological group.*

Proof. Let $\widetilde{G}$ be the completion of the S-bounded topological group G. We prove that $(\widetilde{G}, \widetilde{U}_t)$ is uniformly Scheepers space, where $(\widetilde{G}, \widetilde{U}_t)$ is a completion of the uniform space (G, U_t), U_t - two-sided uniformity. Let $(\widetilde{\alpha}_{U_n} : n \in N) \subset \widetilde{U}$ be an arbitrary sequence of uniform covers, $\widetilde{\alpha}_{U_n} = \{\widetilde{x}\widetilde{\cdot}\widetilde{U}_n\widetilde{\cdot}\widetilde{y} : x \cdot U_n \cdot y \in \alpha_{U_n}\}$, $\widetilde{U}_n = \{F \in \widetilde{G} : F \ni U_n\}$, $\widetilde{\alpha} \in \widetilde{G}$, F - minimal Cauchy filter in (G, U_t). Put $\alpha_{U_n} = \widetilde{\alpha}_{U_n} \wedge \{G\}$. From the construction of completion of uniform spaces we have $(\alpha_{U_n} : n \in N) \subset U_t$. Then there exists a sequence $(\beta_{U_n} : n \in N)$ of finite subfamilies such that for each n, $\beta_{U_n} \subset \alpha_{U_n}$ and for each finite set $M = \{x_1, \ldots, x_m\} \subset G$ there exists $n \in N$ such that $M \subset \bigcup_{x_i \cdot U_n \cdot y_i \in \beta_{U_n}} x_i \cdot U_n \cdot y_i$. Then, there is a sequence $(\widetilde{\beta}_{U_n} : n \in N)$ of finite subfamilies, such that for each $n \in N$, $\widetilde{\beta}_{U_n} \subset \widetilde{\alpha}_{U_n}$, $\widetilde{\beta}_{U_n} = \{\widetilde{x}\widetilde{\cdot}\widetilde{U}_n\widetilde{\cdot}\widetilde{y} : x_i \cdot U_n \cdot y_i \in \beta_{U_n}\}$, where "$\widetilde{\cdot}$" denotes the extension of the algebraic operation "$\cdot$" to the completion, $\widetilde{U}_n = \{F \in \widetilde{G} : F \ni U_n\}$, $\widetilde{x}_i \in \widetilde{G}$, $i = 1, \ldots, m$. Then for each $\widetilde{M} = \{F_{x_1}, \ldots, F_{x_m}\}$ there exists $n \in N$ such that $\widetilde{M} \subset \bigcup_{\widetilde{x}_i\widetilde{\cdot}\widetilde{U}_n\widetilde{\cdot}\widetilde{y}_i \in \widetilde{\beta}_{U_n}} \widetilde{x}_i\widetilde{\cdot}\widetilde{U}_n\widetilde{\cdot}\widetilde{y}_i$, where F_{x_i}, $i = 1, \ldots, m$ is a neighborhood of the points $x_i \in G$, $i = 1, \ldots, m$. □

Theorem 2.6. *The Raikov $\aleph_0$-completion of S-bounded topological group is S-bounded topological group.*

Theorem 2.7. *The product $G \times H$ of S-bounded G and precompact groups H is S-bounded.*

Proof. Let $(\gamma_{U_n} : n \in N)$ be a sequence of uniform covers of $U_l \times V_l$. One may suppose that all elements in each γ_{W_n} are of the form $(x \cdot U_n) \times (y \cdot V_n)$, $x \in G$, $y \in H$ with $x \cdot U_n \in \alpha_{U_n} \in U_l$, $y \cdot V_n \in \beta_{V_n} \in V_l$, $W_n = U_n \times V_n$. For a sequence $(\alpha_{U_n} : n \in N) \subset U_l$ of uniform covers of (G, U_l) there is a sequence $(\alpha^0_{U_n} : n \in N)$ of finite subfamilies of (G, U_l) such that for each $n \in N$, $\alpha^0_{U_n} \subset \alpha_{U_n}$ and for each finite set $F \subset G$ there is $n \in N$ such that $F \subset \alpha^0_{U_n}$, and for a sequence $(\beta_{V_n} : n \in N) \subset V_l$ of uniform covers of (H, V_l) there is a sequence $(\beta^0_{V_n} : n \in N) \subset V_l$ of finite subcover. Let $F' \subset G \times H$ be an arbitrary finite subset of $(G \times H, U_l \times V_l)$. Put $F' = F_G \times F_H$, where $F_G \subset G$ and $F_H \subset H$ are finite subsets. Then there is $n \in N$ such that $F_G \subset \cup\alpha^0_{U_n}$. Since $\beta^0_{V_n}$ is a finite uniform cover of (H, V_l), then $F_H \subset \cup\beta^0_{V_n}$. Then $F' \subset (\cup\alpha^0_{U_n}) \times (\cup\beta^0_{V_n}) = \cup(\alpha^0_{U_n} \times \beta^0_{V_n})$. We put $\alpha^0_{U_n} \times \beta^0_{V_n} = \gamma^0_{W_n}$. Since $\alpha^0_{U_n}$ and $\beta^0_{V_n}$ is finite, then $\gamma^0_{W_n}$ also finite. □

Theorem 2.8. *The product $G \times H$ of S-bounded G and compact groups H is S-bounded.*

3. Conflict of interests

The authors declare that they have no conflict of interests.

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