

Regularization of Systems of Volterra Nonlinear Integral Equations of the Third Kind with the Two Independent Variables

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Abstract

We study a regularization of systems of nonlinear Volterra integral equation of the third kind with continuous data. Regularizing operator is constructed that preserves the property of the Volterra operator. Proven the uniform convergence of the regularized solution to the exact solution.

Keywords: Volterra equations, small parameter, uniform convergence, regularization.

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1. Introduction

The theory of regularization of Volterra integral equations of the third kind is well developed for the one-dimensional case. Study of regularization methods for Volterra integral equations of the third kind are made in [1, 6, 8] in the following cases:

The kernel $K(x, t)$ is integrable with respect to t and this function at $t = x$ vanishes at a finite number of points and the coefficient function degenerates at the initial point of the segment [1, 6];

The kernel of the equation is continuous and the coefficient function degenerates at the end point of the segment [8].

Regularization of Systems of Volterra Linear Integral Equations of the third kind is important because the regularized equation can serve as a theoretical foundation for constructing a numerical solution of equation [4].

In this paper study the construction of the Lavrentjev's type regularization of systems of two-dimensional Volterra integral equation of the third kind with continuous kernel in the case when the kernel is degenerated on the points the diagonal. The results obtained earlier for the scalar case [9] are generalized to the system of equations.

2. Problem Statement

We introduce the following notations:

$\langle \cdot, \cdot \rangle$ is a scalar product in R^n , $\|P\|$ and $\|u\|$ are the norms of the $n \times n$ -matrix P and the n -dimensional vector u , that is $\langle u, v \rangle = u_1 v_1 + \dots + u_n v_n$, $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$, $\|u\| = \sqrt{\langle u, u \rangle}$; $C_n(D)$, is the space of n -dimensional vector with elements in $C(D)$, $D = [0, b] \times [0, c]$, $\|\cdot\|_\infty = \max_D \|\cdot\|$.

Let us consider systems of Volterra integral equations of the third kind

$$P(x, y)u(x, y) + \int_0^x K(x, s, y, u(s, y))ds = \int_0^x \int_0^y N_0(x, y, s, \tau, u(s, \tau))dsd\tau + g(x, y), \quad (2.1)$$

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where $K(x, s, y, u) = K(x, s, y)u + M_0(x, y, s, u)$ and the $n \times n$ -matrix-valued functions $K(x, s, y) = (K_{i,j}(x, s, y))$ and $P(x, y) = (P_{i,j}(x, y))$ are known, $K_{i,j}(x, s, y) \in C(D_1)$, $i, j = \overline{1, n}$, $D_1 = \{(x, s, y) | 0 \leq s \leq x \leq b, 0 \leq y \leq c\}$, $P_{i,j}(x, y) \in C(D)$, $i, j = \overline{1, n}$, $\det P(0, y) = 0$; $g(x, y)$, $M_0(x, y, s, u)$, $N_0(x, y, s, \tau, u)$ are known n -dimensional functions and are continuous, respectively, in the regions D , $D_1 \times R^1$, $D_2 \times R^1$, $D_2 = \{(x, s, y, \tau) | 0 \leq s \leq x \leq b, 0 \leq \tau \leq y \leq c\}$, $g(0, y) = 0$; $u(x, y)$ an unknown n -dimensional function.

Let: (a) $p_i(x, y)$, $i = 1, \dots, n$, be the eigenvalues of the matrix $[P(x, y) + P^*(x, y)]/2$, where $P^*(x, y)$ is the matrix adjoint to $P(x, y)$;

(b) $\lambda_i(x, y)$, $i = 1, \dots, n$, be eigenvalues of the matrix $[G(x, y) + G^*(x, y)]/2$, $G(x, y) = C_0 P(x, y) + K(x, y, x)$, $0 < C_0 = \text{const}$ where $G^*(x, y)$ is the matrix adjoint to $G(x, y)$;

(c) $\mu_i(x, y)$, $i = 1, \dots, n$ be the eigenvalues of the matrix $[P^*(x, y)G(x, y) + G^*(x, y)P(x, y)]/2$.

following conditions:

1) $\min_i p_i(x, y) = p(x, y) \geq 0$, $\lambda(x, y) = \min_i \lambda_i(x, y) \geq d_1 > 0$, $\mu(x, y) = \min_i \mu_i(x, y) \geq C_1 p(x, y) \lambda(x, y)$, $i = 1, \dots, n$, $\|P(x, y)\| \leq C_2 p(x, y)$, $\|G(x, y)\| \leq C_3 \lambda(x, y)$, $0 < C_j = \text{const}$, $j = 1, 2, 3$

and $p(x, y)$ is a nondecreasing function of x in the domain D .

2) $M_0(x, y, x, u) = M_0(x, y, s, 0) = 0$, $N_0(x, y, x, \tau, u) = N_0(x, y, s, \tau, 0) = 0$ and for everyone $t \leq x$

$$\|N_0(x, y, s, \tau, u) - N_0(x, y, s, \tau, \omega) - N_0(t, y, s, \tau, u) + N_0(t, y, s, \tau, \omega)\| \leq L_1(x - t) \|u - \omega\|,$$

$$\|M_0(x, y, s, u) - M_0(x, y, s, \omega) - M_0(t, y, s, u) + M_0(t, y, s, \omega)\| \leq L_2(x - t) \|u - \omega\|,$$

$$0 < L_i = \text{const}, i = 1, 2.$$

3. Regularization

By now only a few works have been published addressing the existence of solutions for third-kind Volterra integral equations [2, 3, 5, 10, 11, 12]. These works have investigated only special classes of third-kind equations with continuous data and weakly singular kernels.

Assume that (2.1) has a continuous solution. Applying operator $I + C_0 J$, where I is identity operator, $Ju = C_0 \int_0^x u(s, y) ds$ is a integral operator, from (2.1) we obtain system of Volterra integral equation of the third kind

$$\begin{aligned} P(x, y)u(x, y) + \int_0^x G(s, y)u(s, y)ds &= \int_0^x M(x, y, s, u(s, y))ds \\ &+ \int_0^x \int_0^y N(x, y, s, \tau, u(s, \tau))dsd\tau + f(x, y), \end{aligned} \quad (3.1)$$

where $M(x, y, s, u) = K(s, y, s)u - K(x, y, s)u - M_0(x, y, s, u) - C_0 \int_s^x K(\nu, y, s, u)d\nu$,

$f(x, y) = g(x, y) + C_0 \int_0^x g(s, y)ds$, $N(x, y, s, \tau, u) = N_0(x, y, s, \tau, u) + C_0 \int_s^x N_0(\nu, y, s, \tau, u)d\nu$.

Consider the system of equation with the small parameter $\varepsilon \in (0, 1)$

$$(I_n + P(x, y))u_\varepsilon(x, y) + \int_0^x G(s, y)u_\varepsilon(s, y)ds = \int_0^x M(x, y, s, u_\varepsilon(s, y))ds +$$

$$+ \int_0^x \int_0^y N(x, y, s, \tau, u_\varepsilon(s, \tau)) ds d\tau + \varepsilon u(0, y) + f(x, y), \quad (3.2)$$

where I_n is the $n \times n$ -unit matrix.

For each $\varepsilon > 0$ the matrix kernel $(-\varepsilon I_n + P(x, y))^{-1}G(s, y)$ has the resolvent [1]

$$R_\varepsilon(x, y, s) = -(\varepsilon I_n + P(x, y))^{-1}X_\varepsilon(x, y, s)G(s, y),$$

where $X_\varepsilon(x, y, s)$ is the matrix-valued Cauchy function of the system

$$X'_{\varepsilon x}(x, y, s) = -G(x, y)(\varepsilon I_n + P(x, y))^{-1}X_\varepsilon(x, y, s),$$

Theorem 3.1. *Let the conditions 1), 2), (a)-(c) take place and system of equation (3.1) has a solution $u(x, y) \in C_n(D)$. Then with $\varepsilon \rightarrow 0$ the solution of system of equation (3.2) uniformly converges to the solution of system of equation (3.1), and the following estimation is fulfilled*

$$\|u_\varepsilon(x, y) - u(x, y)\|_\infty \leq d_2 \varepsilon^{1-\beta} \|u(x, y)\|_\infty + d_3 \omega_u(\varepsilon^\beta), 0 < \beta < 1,$$

where d_2, d_3 are positive constants, $\omega_u(\delta) = \sup\{\|u(x, y) - u(t, y)\|, x, t \in [0, b], y \in [0, c], |x - t| < \delta\}$.

Proof. Using substitution

$$\eta_\varepsilon(x, y) = u_\varepsilon(x, y) - u(x, y) \quad (3.3)$$

from (3.1) and (3.2) we obtain

$$\begin{aligned} (\varepsilon I_n + P(x, y))\eta_\varepsilon(x, y) + \int_0^x G(s, y)\eta_\varepsilon(s, y)ds &= \int_0^x [M(x, y, s, u_\varepsilon(s, y)) - M(x, y, s, u(s, y))]ds \\ &+ \int_0^x \int_0^y [N(x, y, s, \tau, u_\varepsilon(s, \tau)) - N(x, y, s, \tau, u(s, \tau))]dsd\tau + \varepsilon[u(0, y) - u(x, y)]. \end{aligned}$$

This equation using the resolvent $R_\varepsilon(x, y, s)$ of the kernel $[-(\varepsilon I_n + P(x, y))^{-1}G(s, y)]$ is reduced to

$$\begin{aligned} \eta_\varepsilon(x, y) &= \int_0^x R_\varepsilon(x, y, s)(\varepsilon I_n + P(s, y))^{-1} \left\{ \int_0^s [M(s, y, \nu, u_\varepsilon(\nu, y)) - M(s, y, \nu, u(\nu, y))]d\nu \right. \\ &+ \int_0^s \int_0^y [N(s, y, \nu, \tau, u_\varepsilon(\nu, \tau)) - N(s, y, \nu, \tau, u(\nu, \tau))]d\nu d\tau + \varepsilon[u(0, y) - u(s, y)] \Big\} ds \\ &+ (\varepsilon I_n + P(x, y))^{-1} \left\{ \int_0^x [M(x, y, s, u_\varepsilon(s, y)) - M(x, y, s, u(s, y))]ds + \right. \\ &+ \int_0^x \int_0^y [N(x, y, s, \tau, u_\varepsilon(s, \tau)) - N(x, y, s, \tau, u(s, \tau))]dsd\tau + \varepsilon[u(0, y) - u(x, y)] \Big\}. \end{aligned}$$

At this moment we introduce the equivalent alteration.

$$\begin{aligned}
 \eta_\varepsilon(x, y) = & \int_0^x R_\varepsilon(x, y, s)(\varepsilon I_n + P(s, y))^{-1} \left\{ \int_0^s [M(s, y, \nu, u_\varepsilon(\nu, y)) - M(s, y, \nu, u(\nu, y)) \right. \\
 & - M(x, y, \nu, u_\varepsilon(\nu, y)) + M(x, y, \nu, u(\nu, y))] d\nu - \int_s^x [M(x, y, \nu, u_\varepsilon(\nu, y)) - M(x, y, \nu, u(\nu, y))] d\nu \\
 & + \int_0^s \int_0^y [N(s, y, \nu, \tau, u_\varepsilon(\nu, \tau)) - N(s, y, \nu, \tau, u(\nu, \tau)) - N(x, y, \nu, \tau, u_\varepsilon(\nu, \tau)) \\
 & + N(x, y, \nu, \tau, u(\nu, \tau))] d\nu d\tau - \int_s^x \int_0^y [N(x, y, \nu, \tau, u_\varepsilon(\nu, \tau)) - N(x, y, \nu, \tau, u(\nu, \tau))] d\nu d\tau \Big\} ds \\
 & + (\varepsilon I_n + P(x, y))^{-1} X_\varepsilon(x, y, 0) \left\{ \int_0^x [M(x, y, s, u_\varepsilon(s, y)) - M(x, y, s, u(s, y))] ds \right. \\
 & \left. + \int_0^x \int_0^y [N(x, y, s, \tau, u_\varepsilon(s, \tau)) - N(x, y, s, \tau, u(s, \tau))] ds d\tau \right\} + (H_\varepsilon u)(x, y), \quad (3.4)
 \end{aligned}$$

where

$$\begin{aligned}
 (H_\varepsilon u)(x, y) = & \varepsilon \int_0^x R_\varepsilon(x, y, s)(\varepsilon I_n + P(s, y))^{-1} [u(s, y) - u(x, y)] ds + \varepsilon (\varepsilon I_n + P(x, y))^{-1} X_\varepsilon(x, y, 0) \\
 & \times [u(0, y) - u(x, y)].
 \end{aligned}$$

For $X_\varepsilon(x, y, s)$ and $(\varepsilon I_n + P(x, y))^{-1}$ the following estimates hold [1]

$$\|(\varepsilon I_n + P(x, y))^{-1}\| \leq \frac{1}{\varepsilon + p(x, y)},$$

$$\|X_\varepsilon(x, y, s)\| \leq \exp \left(-\frac{\theta}{\sigma^2} \int_s^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu \right),$$

where $\theta = \min(1, C_1)$, $\sigma = \max(1, C_2)$. Since

$$\|M(x, y, \nu, u_\varepsilon) - M(x, y, \nu, u) - M(s, y, \nu, u_\varepsilon) + M(s, y, \nu, u)\| \leq L_M(x - s) \|u_\varepsilon - u\|, \quad s \leq x,$$

$$\|N(x, y, \nu, \tau, u_\varepsilon) - N(x, y, \nu, \tau, u) - N(s, y, \nu, \tau, u_\varepsilon) + N(s, y, \nu, \tau, u)\| \leq L_N(x - s) \|u_\varepsilon - u\|,$$

$s \leq x$, where $L_N = (1 + C_0)L_1$, $L_M = L_2 + L_3 + C_0(L_2 + L_4)$, $L_3 = \max_{D_1} \|K(x, y, s)\|$, $0 < L_4$ is a Lipschitz coefficient of the function $K(x, y, s)$ with respect to x , then of (3.4) we have

$$\|\eta_\varepsilon(x, y)\| \leq E_\varepsilon(x, y) \left\{ L_M \int_0^x \|\eta_\varepsilon(s, y)\| ds + L_N \int_s^x \int_0^y \|\eta_\varepsilon(s, y)\| ds d\tau \right\} + \|(H_\varepsilon u)(x, y)\|,$$

where

$$E_\varepsilon(x, y) = \frac{2C_2}{\varepsilon + p(x, y)} \int_0^x \exp \left(-\frac{\theta}{\sigma^2} \int_s^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu \right) \frac{\lambda(s, y)}{\varepsilon + p(s, y)} (x - s) ds + \\ + \frac{x}{\varepsilon + p(x, y)} \exp \left(-\frac{\theta}{\sigma^2} \int_s^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu \right).$$

Using the condition $\lambda(x) \geq d_1 > 0$ and $\frac{1}{\varepsilon + p(x, y)} \leq \frac{1}{\varepsilon + p(s, y)}$ we obtain

$$E_\varepsilon(x, y) \leq \frac{2C_2}{d_1} \int_0^x \exp \left(-\frac{\theta}{\sigma^2} \int_s^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu \right) \frac{\lambda(s, y)}{\varepsilon + p(s, y)} \int_s^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu ds \\ + \frac{1}{\varepsilon + p(x, y)} \exp \left(-\frac{\theta}{\sigma^2} \int_s^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu \right) d_1^{-1} \int_0^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu.$$

Let $\rho = \frac{\theta}{\sigma^2} \int_s^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu$ and $\xi = \frac{\theta}{\sigma^2} \int_0^x \frac{\lambda(\nu, y)}{\varepsilon + p(\nu, y)} d\nu$. Then

$$E_\varepsilon(x, y) \leq \frac{2C_2\sigma^2}{\theta d_1} \int_0^\infty e^{-\rho} \rho d\rho + \frac{\sigma^2}{\theta d_1} \sup_{\xi \geq 0} [\xi e^{-\xi}] \leq (2C_2 + e^{-1}) \frac{\sigma^2}{\theta d_1}.$$

Then

$$\|\eta_\varepsilon(x, y)\| \leq T_0 L_M \int_0^x \|\eta_\varepsilon(s, y)\| ds + T_0 L_N \int_0^x \int_0^y \|\eta_\varepsilon(s, y)\| ds d\tau + \|(H_\varepsilon u)(x, y)\|_\infty,$$

where $T_0 = (2C_2 + e^{-1})\sigma^2(\theta d_1)^{-1}$.

We apply a modification of the Gronwall–Bellman inequality [9] and estimate by the norm. Then we have

$$\|\eta_\varepsilon(x, y)\|_\infty \leq \exp(bT_0(L_M + cL_N)) \|(H_\varepsilon u)(x, y)\|_\infty.$$

For the operator $(H_\varepsilon u)(x, y)$ the following estimate holds [8]

$$\|(H_\varepsilon u)(x, y)\|_\infty \leq (1 + C_1\sigma^2\theta^{-1}) (2\sigma^2\theta^{-1} \exp(\theta/\sigma^2) \varepsilon^{1-\beta} \|u(x, y)\|_\infty + \omega_u(\varepsilon^\beta)).$$

Hence $\|\eta_\varepsilon(x, y)\|_\infty \rightarrow 0$ at $\varepsilon \rightarrow 0$. Therefore, using (3.3), as $\varepsilon \rightarrow 0$, the function $u_\varepsilon(x, y) \rightarrow u(x, y)$ uniformly. $\square$

4. Case of System with the Approximately Given Right Hand Side

Let us consider the system of equations (2.1) when instead of exactly given right hand side $g(x, y)$ we are given continuous functions $g_\delta(x, y)$ such that $\|g_\delta(x, y) - g(x, y)\| \leq \delta$. Solution of perturbed system of equation

$$P(x, y)u_\delta(x, y) + \int_0^x K(x, s, y, u_\delta(s, y)) ds = \int_0^x \int_0^y N_0(x, y, s, \tau, u_\delta(s, \tau)) ds d\tau + g_\delta(x, y) \quad (4.1)$$

may differ from the exact solution of (2.1) for a significantly large value [3].

Let conditions 1), 2) be fulfilled. Then instead of perturbed system of equation (4.3) let us consider system of equation

$$\begin{aligned} P(x, y)u_\delta(x, y) + \int_0^x G(s, y)u_\delta(s, y)ds &= \int_0^x M(x, y, s, u_\delta(s, y))ds \\ &+ \int_0^x \int_0^y N(x, y, s, \tau, u_\delta(s, \tau))dsd\tau + f_\delta(x, y), \end{aligned} \quad (4.2)$$

where $f_\delta(x, y) = g_\delta(x, y) + C_0 \int_0^x g_\delta(s, y)ds$.

We are going to show that with the proper correlation of ε with the approximation error of initial data δ , and under the above mentioned conditions, the approximate solution of system of equations is the solution of regularized system of equation

$$\begin{aligned} (\varepsilon I_n + P(x, y))u_{\varepsilon, \delta}(x, y) + \int_0^x G(s, y)u_{\varepsilon, \delta}(s, y)ds &= \int_0^x M(x, y, s, u_{\varepsilon, \delta}(s, y))ds \\ &+ \int_0^x \int_0^y N(x, y, s, \tau, u_{\varepsilon, \delta}(s, \tau))dsd\tau + f_\delta(x, y), \end{aligned} \quad (4.3)$$

Using the substitution $\mu_{\varepsilon, \delta}(x, y) = u_{\varepsilon, \delta}(x, y) - u(x, y)$ we obtain system of equation

$$\begin{aligned} (\varepsilon I_n + P(x, y))\mu_{\varepsilon, \delta}(x, y) + \int_0^x G(s, y)\mu_{\varepsilon, \delta}(s, y)ds &= \int_0^x [M(x, y, s, u_{\varepsilon, \delta}(s, y)) - \\ &- M(x, y, s, u(s, y))]ds + \int_0^x \int_0^y [N(x, y, s, \tau, u_{\varepsilon, \delta}(s, \tau)) - N(x, y, s, \tau, u(s, \tau))]dsd\tau - \varepsilon u(x, y) + \\ &+ f_\delta(x, y) - f(x, y). \end{aligned}$$

Using similar methodology as in the proof of Theorem 3.1 we get the estimate

$$\|u_{\varepsilon, \delta}(x, y) - u(x, y)\|_\infty \leq \exp(bT_0(L_M + cL_N))[\|(H_\varepsilon u)(x, y)\|_\infty + (1 + C_1\sigma^2\theta^{-1})\delta/\varepsilon],$$

where

$$\begin{aligned} (H_\varepsilon u)(x, y) &= \varepsilon \int_0^x R_\varepsilon(x, y, s)(\varepsilon I_n + P(s, y))^{-1}[u(x, y) - u(s, y)]ds + \varepsilon(\varepsilon I_n + P(x, y))^{-1} \\ &\times X_\varepsilon(x, y, 0)u(x, y). \end{aligned}$$

From this with $\delta \rightarrow 0$ choosing $\varepsilon = \varepsilon(\delta) \rightarrow 0$ and such that $\delta/\varepsilon \rightarrow 0$ we obtain the convergence of regularized solution $u_{\varepsilon, \delta}(x, y)$ to exact solution $u(x, y)$ of system of equation (3.1).

5. Conclusions

The cases considered allow extending the obtained results to situations where $\det P(b, y) = 0$ and the scalar function $p(x, y)$ in condition 1) is a nonreturning function with respect to x for any $y \in [0, b]$ and $p(b, y) = 0$. This case for systems of one-dimensional linear Volterra equations of the third kind is considered in [7].

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