

On Solvability of Solutions of Nonlocal Problem for Pseudoparabolic Equations with a Small Parameter

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Abstract

A nonlocal problem with Samarskii-Ionkin conditions for a pseudoparabolic equation with a small parameter is considered. The primary focus is on investigating the solvability of the problem in the case where nonlocal conditions of the second kind degenerate into conditions of the first kind at some points in the interval under consideration. A priori estimates are obtained, and a theorem on the unique solvability of the problem in the space of square-summable functions is proved.

Keywords: nonlocal problem, pseudoparabolic equation, small parameter, parabolic equation, solvability.

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1. Introduction

It is known that the issues of liquid filtration in porous media (see [1,2], heat transfer in a heterogeneous medium [4,5] moisture transfer in soils and grounds (see [6], [7, p. 137]) lead to modified diffusion equations, which are pseudoparabolic partial differential equations of the third order. Recently, a significant number of studies have appeared devoted to the study of nonlocal problems. Such problems arise in the mathematical modeling of various processes, such as moisture transfer and heat conductivity, in the study of mathematical biology, control problems, and elsewhere. One class of such problems are nonlocal problems with integral conditions. These conditions describe the behavior of the solution at interior points of the domain in the form of a certain average. The modern stage of development of the theory of such problems began, apparently, with the works of A.V. Bitsadze and A.A. Samarskii [7], published in 1969. Subsequently, in the works of a number of authors, this problem was further developed by N.I. Ionkin [8-11], A.Yu. Mokin [12], N.I. Yurchuk [13] and G. Avalishvili G, M. Avalishvili, B. Miara [14] for second-order partial differential equations. T. B. Benjamin, J. L. Bona, J. J. Mahony [14] and A. Bouziani [18,19] studied various local and nonlocal problems for pseudoparabolic equations were. Local and nonlocal initial-boundary value problems for pseudoparabolic equations are currently being actively studied because applied problems in physics, mechanics, and biology are reduced to such equations and are of great practical and theoretical interest.

The pseudoparabolic equation belongs to the Sobolev class. An important class of Sobolev-type equations are third-order equations of the following types [16,17]:

$$A_0 D_t^l u(x, t) + \sum_{k=0}^{l-1} A_{l-k} D_t^k u(x, t) = f(x, t) \quad (1.1)$$

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where $A_0, A_1, \dots, A_l$ are second-order linear uniformly elliptic differential operators in the spatial variables $x = (x_1, x_2, \dots, x_n)$, D_t^l is a differentiation operator of the l -th order with respect to the variable ($D_t \equiv D_t^1$).

An important class of Sobolev type equations are third-order equations of the following types:

$$D_t u(x, t) - D_t L u(x, t) - M u(x, t) = f(x, t), \quad (1.2)$$

$$D_t^2 u(x, t) - D_t L u(x, t) - M u(x, t) = f(x, t) \quad (1.3)$$

where (L, M) are second-order linear uniformly elliptic differential operators in the spatial variables $x = (x_1, x_2, \dots, x_n)$. Local and inverse problems were studied by B.S. Ablabekov[20].

Following D. Colton, we will call equations of the form (1.2) pseudo-parabolic, since they are regularizing for the Cauchy problem with inverse time for the heat equation, and we will call equations (1.3) pseudohyperbolic.

In this paper, we consider a problem for a pseudoparabolic equation with a nonlocal Ionkin-Samarskii boundary condition and investigate the unique solvability of this problem. Direct and inverse problems for pseudoparabolic equations with local conditions were studied by B.S. Ablabekov, A. Asanov (see [20-26]). In this paper, we consider a problem for a pseudoparabolic equation with a nonlocal Ionkin-Samarskii boundary condition and investigate the unique solvability of this problem.

Problem Statement. Let $T > 0$ be a given fixed number and denote $D_T = \{(x, t) \mid 0 < x < 1, 0 < t \leq T\}$.

In the domain D_T , we consider a non-local initial-boundary value problem for a pseudo-parabolic equation with a small parameter

$$\frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad (x, t) \in D_T, \quad (1.4)$$

satisfying the boundary conditions

$$u(0, t) = u(1, t), \quad u_x(0, t) = 0, \quad 0 \leq t \leq 1, \quad (1.5)$$

and the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1, \quad (1.6)$$

Here $\varepsilon > 0$ is a small parameter. $\varphi(x)$ is a given function.

The classical solution of problem (1.4)–(1.6) is the function $u = u(x, t)$ of the class $C^{(2,1)}(D_T) \cap C(\bar{D}_T)$, which satisfies equation (1.1) with conditions (1.5), (1.6) in the classical sense.

To do this, we first solve the corresponding problem for eigenvalues and eigenfunctions

$$X''(x) + \lambda X(x) = 0, \quad 0 \leq x \leq \pi, \quad X(0) = X(\pi), \quad X'(0) = 0. \quad (1.7)$$

It is required to find all values of λ for which there is a non-trivial solution to the spectral problem (1.4).

In the work [1] it is shown that the problem of solving the spectral problem (1.7) corresponding to problem (1.4)–(1.6) has in $L_2(0, 1)$ a Riesz basis consisting of eigen and associated functions

$$X_0(x) = x, \quad X_{2k-1}(x) = x \cos(2\pi kx), \quad X_{2k}(x) = \sin(2\pi kx), \quad k = 1, 2, 3, \dots \quad (1.8)$$

The sequence of eigenfunctions and associated functions of the problem conjugate to problem (1.4)–(1.6), biorthogonal with $\{X_k(x)\}$,

$$Y_0(x) = 2, \quad Y_{2k-1}(x) = 4 \cos(2\pi kx), \quad Y_{2k}(x) = 4(1-x) \sin(2\pi kx), \quad k = 1, 2, 3, \dots, \quad (1.9)$$

also forms a Riesz basis.

Note that the sequences of functions (1.8) and (1.9) are biorthogonal

$$\langle X_i(x), Y_j(x) \rangle = \int_0^1 X_i(x) Y_j(x) dx = \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \quad (1.10)$$

and form a Riesz basis in the space $L_2(0, 1)$ that is biorthogonal on the interval $(0, 1)$.

Since the set of functions (1.8) forms a basis, any function $\varphi(x) \in L_2(0, 1)$ can be expanded into a biorthogonal series

$$\varphi(x) = \varphi_0 X_0(x) + \sum_{k=1}^{\infty} (\varphi_{2k} X_{2k}(x) + \varphi_{2k-1} X_{2k-1}(x)), \quad (1.11)$$

where $\varphi_0, \varphi_{2k}, \varphi_{2k-1}$ are calculated using the formulas

$$\begin{aligned} \varphi_0 = (\varphi, Y_0(x)) &= \int_0^1 \varphi(x) Y_0(x) dx, \quad \varphi_{2k} = (\varphi(x), Y_{2k}(x)) = \int_0^1 \varphi(x) Y_{2k}(x) dx, \\ \varphi_{2k-1} &= (\varphi(x), Y_{2k-1}(x)) = \int_0^1 \varphi(x) Y_{2k-1}(x) dx. \end{aligned} \quad (1.12)$$

Since $\{Y_k(x)\}_0^\infty$ also forms a basis in $L_2(0, 1)$, the function $\varphi(x)$ is also decomposable into a biorthogonal series

$$\varphi(x) = \tilde{\varphi}_0 Y_0(x) + \sum_{k=1}^{\infty} (\tilde{\varphi}_{2k} Y_{2k}(x) + \tilde{\varphi}_{2k-1} Y_{2k-1}(x)), \quad (1.13)$$

where $\tilde{\varphi}_0, \tilde{\varphi}_{2k}, \tilde{\varphi}_{2k-1}$ are calculated using the formulas

$$\tilde{\varphi}_0 = \int_0^1 \varphi(x) X_0(x) dx, \quad \tilde{\varphi}_{2k} = \int_0^1 \varphi(x) X_{2k}(x) dx, \quad \tilde{\varphi}_{2k-1} = \int_0^1 \varphi(x) X_{2k-1}(x) dx. \quad (1.14)$$

The eigenvalue and eigenfunction problem of the operator is first solved. This problem is linear; therefore, its solution can be represented as a sum of solutions to the problem

$$\frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad (x, t) \in D_T, \quad (1.15)$$

satisfying the boundary conditions

$$u(0, t) = u(\pi, t), \quad 0 \leq t \leq T, \quad \frac{\partial u}{\partial x}(0, t) = 0, \quad 0 \leq t \leq T, \quad (1.16)$$

and the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq \pi, \quad (1.17)$$

and the problem

$$\frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad (x, t) \in D_T, \quad (1.18)$$

satisfying the boundary conditions

$$u(0, t) = u(\pi, t), \quad 0 \leq t \leq T, \quad \frac{\partial u}{\partial x}(0, t) = 0, \quad 0 \leq t \leq T, \quad (1.19)$$

and the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq \pi, \quad (1.20)$$

Problem (1.4)–(1.6) is a singular perturbation of a nonlocal initial-boundary value problem for a parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad (x, t) \in \Omega_T, \quad (1.21)$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l, \quad (1.22)$$

$$u(0, t) = u(1, t), \quad 0 \leq t \leq T, \quad \frac{\partial u}{\partial x}(0, t) = 0, \quad 0 \leq t \leq T. \quad (1.23)$$

2. Existence and Uniqueness of the Classical Solution to the Problem for a Homogeneous Equation

Consider the following problem:

$$Lu = u_t(x, t) - \varepsilon^2 u_{xxt}(x, t) - u_{xx}(x, t) = 0, \quad (x, t) \in D_T, \quad (2.1)$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1, \quad (2.2)$$

$$u(0, t) = u(1, t), \quad u_x(0, t) = 0, \quad 0 \leq t \leq T, \quad (2.3)$$

continuous in the closed domain $\overline{D}_T$. The solution to problem (2.1)–(2.3) will be constructed using the separation of variables method.

The following holds true

Theorem 2.1. *Let $\varphi(x) \in C^2[0, 1]$, $\varphi(0) = \varphi(1)$, $\varphi'(0) = 0$. Then there exists a unique solution to problem (2.1)–(2.3) and it is expressed by the formula*

$$u(x, t) = \varphi_0 X_0(x) + \sum_{k=1}^{\infty} [\varphi_{2k} X_{2k}(x) + \varphi_{2k-1} (X_{2k-1}(x) - 2\pi k t X_{2k}(x))] e^{-\lambda_k t}, \quad (2.4)$$

where

$$\lambda_k = \frac{(2\pi k)^2}{1 + (2\varepsilon\pi k)^2}. \quad (2.5)$$

i.e. it is a classical solution of problem (2.1)–(2.3), where the coefficients φ_{2k} , φ_{2k-1} are calculated using formulas (1.9).

Proof. We seek a solution to problem (2.1)–(2.3) in the form

$$u(x, t) = X(x)T(t). \quad (2.6)$$

Substituting $u(x, t)$ from (2.6) into (2.1) and separating the variables, we arrive at an ordinary differential equation for $T(t)$, which has the form

$$T'(t) + \frac{\lambda}{1 + \varepsilon^2\lambda}T(t) = 0, \quad (2.7)$$

and the spectral problem for the eigenvalues λ and eigenfunctions $X(x)$

$$X''(x) + \lambda X(x) = 0, \quad 0 \leq x \leq 1, \quad (2.8)$$

$$X'(0) = X'(1), \quad X(0) = 0. \quad (2.9)$$

In accordance with (2.6), we introduce the functions

$$u_0(x, t) = \varphi_0 X_0(x),$$

$$u_k(x, t) = X_k(x)T_k(t) \equiv [\varphi_{2k}X_{2k}(x) + \varphi_{2k-1}(X_{2k-1}(x) - 2\pi ktX_{2k}(x))]e^{-\lambda_k t}, \quad k = 1, 2, \dots$$

In accordance with (2.6), we introduce the functions and note that the series (2.4) is the sum of the series $u_k(x, t)$, $k = 0, 1, 2, \dots$, where

$$u_0(x, t) = \varphi_0 X_0(x), \quad u_k(x, t) = [\varphi_{2k}X_{2k} + \varphi_{2k-1}(X_{2k-1} - 2\sqrt{\lambda_k}tX_{2k})]e^{-\omega_k t}, \quad k = 1, 2, \dots \quad (2.10)$$

$$\begin{aligned} \frac{\partial u_k(x, t)}{\partial t} &= -\omega_k [\varphi_{2k}X_{2k} + \varphi_{2k-1}(X_{2k-1} - 2\sqrt{\lambda_k}tX_{2k})]e^{-\omega_k t} + \\ &+ \varphi_{2k-1}2\sqrt{\lambda_k}X_{2k}e^{-\omega_k t}, \quad k = 1, 2, \dots \end{aligned}$$

$$\frac{\partial^2 u_k(x, t)}{\partial x^2} = [-\varphi_{2k}\lambda_k X_{2k} + \varphi_{2k-1}(-\lambda_k X_{2k-1} + 2\lambda_k^{3/2}tX_{2k})]e^{-\omega_k t}, \quad k = 1, 2, \dots$$

$$\begin{aligned} \frac{\partial^3 u_k(x, t)}{\partial x^2 \partial t} &= -\omega_k [-\varphi_{2k}\lambda_k X_{2k} + \varphi_{2k-1}(-\lambda_k X_{2k-1} + 2\lambda_k^{3/2}tX_{2k})]e^{-\omega_k t} + \\ &+ \varphi_{2k-1}(-\lambda_k X_{2k-1} + 2\lambda_k^{3/2}t^2 X_{2k})e^{-\omega_k t}, \quad k = 1, 2, \dots \end{aligned}$$

From the continuity of the function $\varphi(x)$, it follows that $|\varphi(x)| \leq M$.

$$|\varphi_{2k}| = |(\varphi(x), Y_{2k}(x))| = |(\varphi(x), 4(1-x)\sin(2\pi kx))| \leq 4M,$$

$$|\varphi_{2k-1}| = |(\varphi(x), Y_{2k-1}(x))| = |(\varphi(x), 4\cos(2\pi kx))| \leq 4M,$$

Using these estimates, as well as $|\omega_k| \leq 1$, we obtain

$$|u_k(x, t)| \leq \left[4M + 4M(1+t)\sqrt{\lambda_k} \right] \exp(-\omega_k t_0) \leq 8M(1+t)\sqrt{\lambda_k} \exp(-\omega_k t_0) \leq M_1 \lambda_k^{1/2},$$

$$k = 1, 2, \dots$$

From the construction it follows that the functions $u_k(x, t)$ for any k satisfy equation (2.1) and boundary conditions (2.3). The same (2.4) is also true for any linear combination of functions (2.9), as well as series (2.4), provided that it converges uniformly in the closed domain $\bar{D}_T$. $\square$

3. The Fourier Method for Solving a Non-Homogeneous Equation

Let us assume that we need to find a classical solution to the problem

$$Lu = u_t(x, t) - u_{xxt}(x, t) - u_{xx}(x, t) = f(x, t), \quad (x, t) \in D_T, \quad (3.1)$$

$$u(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (3.2)$$

$$u(0, t) = u(1, t), \quad u_x(0, t) = 0, \quad 0 \leq t \leq T, \quad (3.3)$$

Theorem 3.1. *If $f(x, t) \in C^{(2,1)}(D_T)$, $f_{xx}(x, t) \in L_1(0, 1, C[0, T])$ and $f(0, t) = 0$, $f_x(0, t) = f_x(1, t)$, $f_{xx}(0, t) = f_{xx}(1, t)$, then the solution to problem (3.1) - (3.3) exists and is unique. This solution has the form*

$$\begin{aligned} u(x, t) = & \left[\varphi_0 + \int_0^t f(\tau) h_0(\tau) d\tau \right] X_0(x) + \sum_{k=1}^{\infty} \left[\varphi_{2k} e^{-\omega_k t} + \int_0^t e^{-\omega_k(t-\tau)} f(\tau) h_{2k}(\tau) d\tau \right] X_{2k}(x) + \\ & + \sum_{k=1}^{\infty} \left[\varphi_{2k-1} e^{-\omega_k t} - 4\pi k \varphi_{2k} t e^{-\omega_k t} + \int_0^t e^{-\omega_k(t-\tau)} f(\tau) h_{2k-1}(\tau) d\tau \right] X_{2k-1}(x) - \\ & - 4\pi \sum_{k=1}^{\infty} k \left[\int_0^t e^{-\omega_k(t-\tau)} (t-\tau) f(\tau) h_{2k-1}(\tau) d\tau \right] X_{2k-1}(x). \end{aligned} \quad (3.4)$$

Proof. We seek the solution to problem (3.1) - (3.3) in the form of a series

$$u(x, t) = \sum_{k=0}^{\infty} T_k(t) X_k(x). \quad (3.5)$$

in terms of the eigenfunctions of the corresponding homogeneous problem. Having expanded the functions $f(x, t)$ in a Fourier series

$$f(x, t) = f_0 X_0(x) + \sum_{k=1}^{\infty} (f_{2k}(t) X_{2k}(x) + f_{2k-1}(t) X_{2k-1}(x)), \quad (3.6)$$

where

$$f_0(t) = \int_0^1 f(x, t) Y_0(x) dx, \quad f_{2k}(t) = \int_0^1 f(x, t) Y_{2k}(x) dx, \quad f_{2k-1}(t) = \int_0^1 f(x, t) Y_{2k-1}(x) dx. \quad (3.7)$$

$$T_0'(t) + \frac{\lambda_0^2}{1 + (\varepsilon^2 \lambda_0^2)} T_0(t) = \frac{1}{1 + (\varepsilon^2 \lambda_0^2)} f_0(t), \quad T_0(0) = 0, \quad (3.8)$$

$$T_{2k}'(t) + \frac{\lambda_k^1}{1 + (\varepsilon \lambda_k^1)^2} T_{2k}(t) = \frac{1}{1 + (\varepsilon \lambda_k^1)^2} f_{2k}(t), \quad T_{2k}(0) = 0, \quad (3.9)$$

$$T_{2k-1}'(t) + \frac{\lambda_k^2}{1 + (\varepsilon^2 \lambda_k^2)^2} T_{2k-1}(t) = \frac{1}{1 + (\varepsilon^2 \lambda_k^2)^2} f_{2k-1}(t), \quad T_{2k-1}(0) = 0, \quad (3.10)$$

The solution to problem (3.8)-(3.10) exists, is unique, and is expressed explicitly

$$\begin{aligned} T_0(t) &= \frac{1}{1 + (\varepsilon^2 \lambda_0^2)} \int_0^t f_0(\tau) \exp(-\omega_0^2(t - \tau)) d\tau, \\ T_{2k}(t) &= \frac{1}{1 + (\varepsilon^2 \lambda_{2k}^2)} \int_0^t f_{2k}(\tau) \exp(-\omega_{2k}^2(t - \tau)) d\tau, \\ T_{2k-1}(t) &= \frac{1}{1 + (\varepsilon \lambda_{2k-1})^2} \int_0^t f_{2k-1}(\tau) \exp(-\omega_{2k-1}^2(t - \tau)) d\tau. \end{aligned} \quad (3.11)$$

Substituting (3.11) into the expressions for $T_k(t)$ in (3.5), we obtain a formal solution to problem (3.1) - (3.3):

$$\begin{aligned} u(x, t) &= \frac{1}{1 + (\varepsilon^2 \lambda_0^2)} \int_0^t f_0(\tau) \exp(-\omega_0^2(t - \tau)) d\tau X_k(x) + \\ &\sum_{k=1}^{\infty} \frac{1}{1 + (\varepsilon^2 \lambda_{2k}^2)} \left(\int_0^t f_{2k}(\tau) \exp(-\omega_{2k}^2(t - \tau)) d\tau \right) X_{2k}(x) + \\ &+ \sum_{k=1}^{\infty} \frac{1}{1 + (\varepsilon^2 \lambda_{2k-1}^2)} \left(\int_0^t f_{2k-1}(\tau) \exp(-\omega_{2k-1}^2(t - \tau)) d\tau \right) X_{2k-1}(x). \end{aligned} \quad (3.12)$$

From here, changing the order of summation and integration, using (3.12), we write it in the form (3.4). The conditions imposed on $f(x, t)$ imply the convergence of the series

$$\sum_{k=0}^{\infty} w_k, \quad \sum_{k=0}^{\infty} w_{kt}(x), \quad \sum_{k=0}^{\infty} w_{kxx}(x), \quad \sum_{k=0}^{\infty} w_{kxxx}(x).$$

Theorem 3.1. is proven: □

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