

On One Class of Nonlinear Fredholm Integral Equations of the Third Kind with Multipoint Singularities on the Semi-Axis

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Abstract

On the basis of the new approach it is shown that solving a class of nonlinear Fredholm integral equations of the third kind with multipoint singularities on the semi-axis are equivalent to solving systems of nonlinear algebraic equations. The questions of existence and uniqueness of the solution for these nonlinear integral equations are studied.

Keywords: Solution, nonlinear integral equation, Fredholm, third kind, multipoint singularities, the semi-axis, equivalence, system of nonlinear algebraic equations.

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1. Introduction

Many questions for integral equations were investigated in [1-15]. The survey of results on Volterra integral equations of the second kind is given in [1]. Regularizing operators according to M.M.Lavrentiev were constructed in [2] to solve linear integral Fredholm equations of the first kind. The existence of a multiparameter family of solutions was proved for linear Volterra integral equations of the first and third kind with smooth kernels in [3]. In [4-6], uniqueness theorems were proved for systems of nonlinear Volterra integral equations of the first and third kind and for systems of linear integral Fredholm equations of the third kind and regularizing operators according to M.M.Lavrentiev were constructed. In [7], the issues of uniqueness and existence of solutions of nonclassical linear integral Volterra equations of the first kind and their application in various applied problems are investigated. In [8-9], the issues of uniqueness and existence of solutions to systems of nonlinear Volterra integral equations of the first kind and their application in inverse problems are investigated. In [10], based on a new approach, the issues of uniqueness, non-uniqueness and the existence of solutions for one class of systems of linear integral Fredholm equations of the third kind were studied. In [11-12], based on a modification of the approach proposed in [10], the issues of uniqueness, non-uniqueness and the existence of solutions for scalar and systems of nonlinear Fredholm integral equations of the third kind with multipoint singularities on the segment were investigated. In [13-14], based on a modification of the approach proposed in [10-12], various issues of systems of linear integral Fredholm equations of the third kind with multipoint singularities on the axis and on the semi-axis were studied.

In this paper, based on the approach proposed in [10-12], the questions of the existence and uniqueness of a solution in the space of continuous functions on a semi-axis for one class of the nonlinear Fredholm integral equations of the third kind with multipoint singularities on the semi-axis are investigated.

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2. Preliminaries and Definitions

Denote by $C[a, \infty)$ is a space of all continuous functions on $[a, \infty)$. We will denote by $L_1[a, \infty)$ the space of all integrable functions in $[a, \infty)$. It is known that the space $C[a, \infty)$ is not a normalized space. But the space $L_1[a, \infty)$ is the Banach space.

3. One Class of the Nonlinear Fredholm Integral Equations of the Third Kind

Consider the family of nonlinear integral equations of the third kind

$$p(x)u(x) = \int_a^\infty \sum_{i=1}^m a_i(x)k_i(s, u(s)) ds + f(x), \quad x \in [a, \infty), \quad (3.1)$$

where $p(x), a_j(x), f(x)$ are known continuous functions on $[a, \infty)$ ($j = 1, \dots, m$), $k_i(x, u)$ is a known continuous function on $[a, \infty) \times \mathbb{R}$, $i = 1, 2, \dots, m$, $u(x)$ is an unknown continuous function on $[a, \infty)$, $p(x_l) = 0$, $x_l \in [a, \infty)$, $l = 1, 2, \dots, k$.

Throughout this paper we assume that

$$p(x) = \prod_{l=1}^k p_l(x), \quad p_l(x_l) = 0, \quad x_l \in [a, \infty), \quad p_l(x) \in C[a, \infty), \quad (3.2)$$

$p_l(x) \neq 0$ for all $x \in [a, \infty)$, $x \neq x_l$, $l = 1, \dots, k$.

Assume that the following conditions hold:

(a) For all $l = 1, \dots, k$ and $j = 1, \dots, m$, $a_{l,j}(x)$ are continuous functions on $[a, \infty)$, where

$$a_{0,j}(x) = a_j(x), \quad a_{l,j}(x) = \frac{1}{p_l(x)} [a_{l-1,j}(x) - a_{l-1,j}(x_l)], \quad x \in [a, \infty);$$

(b) For all $l = 1, \dots, k$, $f_l(x)$ functions are continuous functions on $[a, \infty)$, where

$$f_0(x) = f(x), \quad f_l(x) = \frac{1}{p_l(x)} [f_{l-1}(x) - f_{l-1}(x_l)], \quad x \in [a, \infty);$$

(c) For all $i = 1, \dots, m$ and for all $u(x) \in C[a, \infty)$ the function $k_i(x, u(x))$ is an element from the space $L_1[a, \infty)$.

Let

$$c_i = \int_a^{x_l} k_i(s, u(s)) ds, \quad i = 1, 2, \dots, m. \quad (3.3)$$

Then from (1) we get

$$p(x)u(x) = \sum_{j=1}^m a_j(x)c_j + f(x), \quad x \in [a, \infty), \quad (3.4)$$

where c_j - unknown numbers, $j = 1, 2, \dots, m$. Assuming $x = x_1$, from (3.4) we have

$$\sum_{j=1}^m a_j(x_1)c_j + f(x_1) = 0. \quad (3.5)$$

Subtracting (3.5) from (3.4) we get

$$p(x)u(x) = \sum_{j=1}^m [a_j(x) - a_j(x_1)]c_j + f(x) - f(x_1), \quad x \in [a, \infty). \quad (3.6)$$

If $k \geq 2$, then by virtue of (3.2), the conditions a) and b) from (3.6), we obtain

$$\prod_{i=2}^k p_i(x) u(x) = \sum_{j=1}^m a_{1,j}(x)c_j + f_1(x), \quad x \in [a, \infty). \quad (3.7)$$

Assuming $x = x_2$, from (3.7) we have

$$\sum_{j=1}^m a_{1,j}(x_2)c_j + f_1(x_2) = 0. \quad (3.8)$$

Subtracting (3.8) from (3.7) we get

$$\prod_{i=2}^k p_i(x) u(x) = \sum_{j=1}^m [a_{1,j}(x) - a_{1,j}(x_2)]c_j + f_1(x) - f_1(x_2) = 0, \quad x \in [a, \infty). \quad (3.9)$$

If $k \geq 3$, then by the conditions a) and b) from (3.9), we obtain

$$\prod_{i=3}^k p_i(x) u(x) = \sum_{j=1}^m a_{2,j}(x)c_j + f_2(x), \quad x \in [a, \infty). \quad (3.10)$$

Thus, we continue this process to determine the unknown numbers $c_1, c_2, \dots, c_m$. from (3.10), we obtain the following systems

$$\begin{cases} \sum_{j=1}^m a_j(x_1)c_j + f(x_1) = 0, \\ \sum_{j=1}^m a_{l,j}(x_{l+1})c_j + f_l(x_{l+1}) = 0, \quad l = 1, 2, \dots, k-1. \end{cases} \quad (3.11)$$

Taking into account (3.11) from (3.10) we have

$$u(x) = \sum_{j=1}^m a_{k,j}(x)c_j + f_k(x), \quad x \in [a, \infty). \quad (3.12)$$

Substituting (3.12) into (3.3) we get

$$c_i = \int_a^b k_i \left(s, \sum_{j=1}^m a_{k,j}(s)c_j + f_k(s) \right) ds, \quad i = 1, 2, \dots, m. \quad (3.13)$$

Thus, we have a system of nonlinear algebraic equations (3.11) and (3.13), which defines c_i , $i = 1, 2, \dots, m$.

Theorem 3.1. *Let the conditions (3.2), a), b) and c) be satisfied. Then:*

1. *If the system of nonlinear algebraic equations (3.11) and (3.13) has no solutions, then the system of nonlinear integral equations (3.1) has no solutions in the space $C[a, \infty)$;*

2. If the system of nonlinear algebraic equations (3.11) and (3.13) has a unique solution $c_1, c_2, \dots, c_m$, then the system of nonlinear integral equations (3.1) has a unique solution in the space $C[a, \infty)$. This solution is determined by the formula (3.12);
3. If the system of nonlinear algebraic equations (3.11) and (3.13) have a finite number r of solutions $c_{1,q}, c_{2,q}, \dots, c_{m,q}$, where $q = 1, 2, \dots, r$, then the system of nonlinear integral equations (3.1) in the space $C[a, \infty)$ have a finite number r of solutions $u_1(t), u_2(t), \dots, u_r(t)$ defined by the formula:

$$u_q(t) = \sum_{j=1}^m a_{k,j}(x) c_{j,q} + f_k(x), \quad t \in [a, \infty), \quad q = 1, 2, \dots, r. \quad (3.14)$$

- Proof.* 1. It is evident that if systems (3.11) and (3.13) have no solutions, then the system of nonlinear integral equations (3.1) has no solutions in the space $C[a, \infty)$.
2. Let systems (3.11) and (3.13) have a unique solution. Then the system of nonlinear integral equations (3.1) has a unique solution in the space $C[a, \infty)$. This solution is determined by the formula (3.12), where $c_1, c_2, \dots, c_m$ is the only solution of the system (3.11) and (3.13).
 3. Let the system of nonlinear algebraic equations (3.11) and (3.13) have a finite number r of solutions $c_{1,q}, c_{2,q}, \dots, c_{m,q}$, where $q = 1, 2, \dots, r$. Then by virtue of (3.12) the system of nonlinear integral equations (3.1) in the space $C[a, \infty)$ there are r solutions defined by the formula (3.14). The theorem has been proved. □

Example. Consider the nonlinear integral equations (3.1), where

$$a = 0, \quad m = 2, \quad k = 2, \quad p(x) = x(x-1), \quad a_1(x) = x, \quad a_2(x) = 1,$$

$$k_1(s, u) = \frac{1}{(s+1)^2 + u^2}, \quad k_2(s, u) = \frac{q}{(s+3)^2 + u^2}, \quad f(x) = \alpha x^2 + \beta x + \gamma, \quad (3.15)$$

where q, α, β, γ are parameters.

In this example, condition c) is satisfied.

Let's introduce the following notation

$$c_1 = \int_0^\infty \frac{ds}{(s+1)^2 + u^2(s)}, \quad c_2 = \int_0^\infty \frac{q ds}{(s+3)^2 + u^2(s)}. \quad (3.16)$$

In this case

$$x_1 = 0, \quad x_2 = 1, \quad p_1(x) = x, \quad p_2(x) = x-1, \quad a_{1,1}(x) = 1,$$

$$a_{1,2}(x) = 0, \quad a_{2,1}(x) = 0, \quad a_{2,2}(x) = 0, \quad f_1(x) = \alpha x + \beta, \quad f_2(x) = \alpha x. \quad (3.17)$$

Here condition c) is fulfilled. Since for all $u(x) \in C[0, \infty)$:

$$k_1(x, u(x)) = \frac{1}{(x+1)^2 + u^2(x)} \leq \frac{1}{(x+1)^2} \in L_1[0, \infty),$$

$$k_2(x, u(x)) = \frac{q}{(x+3)^2 + u^2(x)} \leq \frac{q}{(x+3)^2} \in L_1[0, \infty).$$

Further considering (3.15),(3.16),(3.17) the system (3.11) and formula (3.12) are written as

$$\begin{cases} c_2 + \gamma = 0, \\ c_1 + \alpha + \beta = 0, \end{cases} \quad (3.18)$$

$$u(x) = \alpha x, \quad \forall x \in [0, \infty). \quad (3.19)$$

The first case: Let $\alpha > 0$. Then substituting (3.19) into (3.16) we have

$$c_1 = \frac{1}{\alpha} \left(\frac{\pi}{2} - \tan^{-1} \frac{1}{\alpha} \right), \quad c_2 = \frac{q}{\alpha} \left(\frac{\pi}{2} - \tan^{-1} \frac{3}{\alpha} \right). \quad (3.20)$$

Considering (3.20) from (3.18), we obtain

$$\begin{cases} \gamma = -\frac{q}{\alpha} \left(\frac{\pi}{2} - \tan^{-1} \frac{3}{\alpha} \right), \\ \beta = -\alpha - \frac{1}{\alpha} \left(\frac{\pi}{2} - \tan^{-1} \frac{1}{\alpha} \right). \end{cases} \quad (3.21)$$

Then by virtue of (3.21) the following cases take place:

1. If $\alpha > 0, \beta \neq -\alpha - \frac{1}{\alpha} \left(\frac{\pi}{2} - \tan^{-1} \frac{1}{\alpha} \right)$, then the nonlinear integral equations (3.1) have no solutions in the space $C[0, \infty)$.
2. If $\alpha > 0, \gamma \neq -\frac{q}{\alpha} \left(\frac{\pi}{2} - \tan^{-1} \frac{3}{\alpha} \right)$, then the nonlinear integral equations (3.1) have no solutions in the space $C[0, \infty)$.
3. If condition (3.21) is met and $\alpha > 0$, then the nonlinear integral equations (3.1) in the space $C[0, \infty)$ have a unique solution:
 $u(x) = \alpha x, \quad \forall x \in [0, \infty).$

The second case: Let $\alpha = 0$. Then substituting (3.19) into (3.16) we have

$$c_1 = 1, \quad c_2 = \frac{q}{3}. \quad (3.22)$$

Considering (3.22) from (3.18), we obtain

$$\beta = -1, \quad \gamma = -\frac{q}{3}. \quad (3.23)$$

Then by virtue of (3.23) the following cases take place:

4. If $\alpha = 0, \beta \neq -1$, then the nonlinear integral equations (3.1) have no solutions in the space $C[0, \infty)$.
5. If $\alpha = 0, \gamma \neq -\frac{q}{3}$, then the nonlinear integral equations (3.1) have no solutions in the space $C[0, \infty)$.

6. If $\alpha = 0$, $\beta = -1$, $\gamma = -\frac{q}{3}$, then the nonlinear integral equations (3.1) in the space $C[0, \infty)$ have a unique solution:

$$u(x) = 0, \quad \forall x \in [0, \infty).$$

The third case: Let $\alpha < 0$. Then substituting (3.19) into (3.16) we have

$$c_1 = \frac{1}{\alpha} \left(-\frac{\pi}{2} - \tan^{-1} \frac{1}{\alpha} \right), \quad c_2 = \frac{q}{\alpha} \left(-\frac{\pi}{2} - \tan^{-1} \frac{3}{\alpha} \right). \quad (3.24)$$

Considering (3.24) from (3.18), we obtain

$$\begin{cases} \gamma = \frac{q}{\alpha} \left(\frac{\pi}{2} + \tan^{-1} \frac{3}{\alpha} \right), \\ \beta = -\alpha + \frac{1}{\alpha} \left(\frac{\pi}{2} + \tan^{-1} \frac{1}{\alpha} \right). \end{cases} \quad (3.25)$$

Then by virtue of (3.25) the following cases take place:

7. If $\alpha < 0$, $\beta \neq -\alpha + \frac{1}{\alpha} \left(\frac{\pi}{2} + \tan^{-1} \frac{1}{\alpha} \right)$, then the nonlinear integral equations (3.1) have no solutions in the space $C[0, \infty)$.

8. If $\alpha < 0$, $\gamma \neq \frac{q}{\alpha} \left(\frac{\pi}{2} + \tan^{-1} \frac{3}{\alpha} \right)$, then the nonlinear integral equations (3.1) have no solutions in the space $C[0, \infty)$.

9. If condition (3.25) is met and $\alpha < 0$, then the nonlinear integral equations (3.1) in the space $C[0, \infty)$ have a unique solution:

$$u(x) = \alpha, \quad \forall x \in [0, \infty).$$

4. Conclusion

It is known that different applied problems have been studied by reducing to Fredholm integral equations of the second kind. But applied problems reduced to Fredholm integral equations of the third kind have been little studied. Since Fredholm integral equations of the third kind have been little studied. Here, based on the research approach proposed in [10-13], qualitative studies of nonlinear Fredholm integral equations of the third kind on the semi-axis have been carried out.

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