

Regularization of a Loaded Inverse Problem of the Korteweg-de Vries Type in an Unbounded Domain

A.M. Alybaev^{*1}

¹ *Jusup Balasagyn Kyrgyz National University, Bishkek, Kyrgyzstan*

Abstract

This paper considers an inverse problem for a loaded differential operator of the Korteweg-de Vries type in an unbounded domain. It is shown that the problem reduces to a degenerate ill-posed nonlinear Volterra integral equation of the first kind of a special form.

The study is carried out using the Fourier transform and a regularization method in specially introduced functional spaces. The regularizability of the inverse problem is investigated in a generalized sense.

It is noted that such loaded inverse problems of the Korteweg-de Vries type, leading to ill-posed Volterra integral equations of the first kind, arise in wave theory, in particular in the study of propagation processes of dispersive loaded waves and related phenomena.

Keywords: Volterra integral equation of the first kind (VIE-1), inverse problem (IP), regularization method (RM), regularizability in the generalized sense, Fourier transform (FT), Dirac function (DF).

2010 MSC: 517.9

1. Introduction

The paper investigates a coefficient IP for a loaded Korteweg-de Vries type equation in an unbounded domain arising in wave theory and in problems related to the propagation of dispersive loaded waves and related processes [1,2,3,4,5,6,7,8].

It is established that the considered problem leads to a degenerate ill-posed VIE-1. The study is carried out using the apparatus of the FT and RM in the generalized sense.

Let the IP be given by:

$$U_t + \lambda U(U_x(t, x_0))^2 + U_{x^3} = \varphi(x)(J\theta)(t), \quad \forall (t, x) \in \bar{D}, \quad (D = (0, T) \times \mathbb{R}), \quad (1.1)$$

$$U(0, x) = \varphi(x), \quad \forall x \in \mathbb{R}, \quad (1.2)$$

$$(U_t + U_{x^3})|_{x=0} = g(t), \quad \forall t \in [0, T], \quad (1.3)$$

where

$$J\theta = \sum_{i=0}^1 \left(\lambda_i \int_0^t K_0(t, s) \theta(s) ds \right)^{i+1}, \quad (\lambda_0 = 1), \quad (1.4)$$

and $\lambda, \lambda_1, \varphi(x), g(t), K_0(t, s)$ are the known functions satisfying the conditions:

^{*}Corresponding author: anarbek.alymbaev@mail.ru

- a)** $\lambda < 1$, $0 < \lambda_1$; $\lambda, \lambda_1 = \text{const}$; $\varphi(x) \in C^3(\mathbb{R})$; $g(t) \in C^1[0, T]$.
- b)** $K_0(t, s) \in C(D_0) \cap \text{Lip}(t \mid L_{K_0})$, $(D_0 = \{(t, s) : 0 \leq s \leq t \leq T\})$, $K_0(t, s) \geq 0$,
- $$K_0(t, 0) \equiv q = \text{const}; \quad |K_0(t, s)| \leq C_{01} = \text{const}.$$

Since the unknown value is a vector function: $P = (U, \theta)$ from a vector space $\tilde{G}_{[\tilde{W}(D), Z^1(0, T)]}^{1,1}(D) = \{(U(t, x), \theta(t)) : U \in \tilde{W}^1(D), \theta \in Z^1(0, T)\}$ endowed with the norm:

$$\begin{cases} \|P\|_{\tilde{G}_{[\tilde{W}(D), Z^1(0, T)]}^{1,1}(D)} = \|U\|_{\tilde{W}^1(D)} + \|\theta\|_{Z^1(0, T)}, \\ \|U\|_{\tilde{W}^1(D)} = \|U\|_{C^{0,3}(D)} + \|U_t\|_1, \\ \|U_t\|_1 = \sup_{\mathbb{R}} \int_0^T |U_t(t, x)| dt. \end{cases} \quad (1.5)$$

Here $Z^1(0, T)$ is a space containing all elements $L^1[0, T]$, as well as nonnegative elements $z(t)$, associated with the DF, provided that there exists a test function satisfying the condition: $g(0) = 0$, $t_0 < g, z \geq 0$.

Then it is required to prove the regularizability of the original IP in the generalized sense in the indicated space.

Note that from the studied IP (1.1), (1.2), (1.3) there arises a degenerate ill-posed VIE-1 with a nonnegative solution associated with the DF namely:

$$J\theta \equiv \int_0^t K_0(t, s)\theta(s) ds + \lambda_1 \left(\int_0^t K_0(t, s)\theta(s) ds \right)^2 = F(t), \quad (1.6)$$

Moreover, the free term of this integral equation satisfies condition:

- c)** $F(t) \in C[0, T] \cap \text{Lip}(t \mid L_F)$; $F(t) > 0$, $(F(0) \neq 0)$; $|F(t)| \leq C_{02}$.

Hence, under condition (c), VIE-1 (1.6) is ill-posed in $C[0, T]$. Therefore, if we assume

$$\theta = \alpha\delta(t) + V(t), \quad (1.7)$$

then from VIE-1 (1.6) it follows that

$$\alpha q + \int_0^t K_0(t, s)V(s) ds + \lambda_1 \left(\alpha q + \int_0^t K_0(t, s)V(s) ds \right)^2 = F(t). \quad (1.8)$$

Further, in order to determine the unknown constant, taking into account with $t = 0$ from in (1.3), we obtain

$$\alpha q + \lambda_1(\alpha q)^2 = F(0),$$

or the quadratic equation:

$$\lambda_2 \alpha^2 + \alpha q - F(0) = 0, \quad (\lambda_2 = \lambda_1 q^2, \lambda_1 > 0). \quad (1.9)$$

With respect to (1.9), the condition

$$\tilde{D} = q^2 + 4F(0)\lambda_2 > 0, \quad (F(0) > 0, \lambda_2 > 0),$$

holds; hence we obtain

$$\alpha = \frac{-q + \sqrt{\tilde{D}}}{2\lambda_2} = q_0 > 0, \quad (1.10)$$

Since, by the assumption, VIE-1 (1.6) has a nonnegative solution, the root

$$\alpha = \frac{-q - \sqrt{\tilde{D}}}{2\lambda_2} < 0. \quad (1.11)$$

Since, by the assumption, VIE-1 (1.6) has a nonnegative solution, the root $\alpha = \frac{-q - \sqrt{\tilde{D}}}{2\lambda_2} < 0$, is not taken into account. Consequently, substituting (1.11) into (1.7) we obtain

$$qq_0 + \int_0^t K_0(t, s)V(s) ds + \lambda_1 \left(qq_0 + \int_0^t K_0(t, s)V(s) ds \right)^2 = F(t),$$

or

$$\begin{cases} \int_0^t K_0(t, s)V(s) ds + 2\lambda_1 qq_0 \int_0^t K_0(t, s)V(s) ds + \lambda_1 \left(\int_0^t K_0(t, s)V(s) ds \right)^2 = F(t) - F_0, \\ qq_0 + \lambda_1 (qq_0)^2 = \bar{F}_0 = \text{const}, \quad (\bar{F}_0 > 0); \quad t = 0 : F(t)|_{t=0} - \bar{F}_0 = 0. \end{cases} \quad (1.12)$$

Therefore, solving equation (1.12) on the basis of (1.7), we obtain a solution of equation (1.6). However, since VIE-1 (1.12) with a kernel of type (b) has not yet been investigated in a Banach space, algorithm (1.7) does not provide answers to the questions posed for VIE-1 (1.6). Thus, the regularizability of equation (1.6) remains open. To prove the regularizability of VIE-1 (1.6), this section proposes a variant of the RM in MP in $Z^1(0, T)$.

It is known that if $\lambda_1 = 0$, then from (1.6) we obtain the linear VIE-1:

$$\int_0^t K_0(t, s)\theta(s) ds = F(t) \quad (1.13)$$

with conditions (b, c) imposed on the known functions. The obtained results for VIE-1 (1.6) are also applicable to VIE-1 (1.13). Moreover, in the case: $K_0(t, s) \equiv 1$, $F(t) \equiv 1$, the equation (1.13) implies the VIE-1:

$$\int_0^t \theta(s) ds = 1, \quad (1.14)$$

which admits a solution of the form:

$$\theta = \delta(t) = \begin{cases} 0, & t > 0, \\ \infty, & t = 0. \end{cases}$$

2. Integration of the IP (1.1), (1.2), (1.3)

For this purpose, we use the FT, assuming

$$\begin{cases} U(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} z(t, 0, \omega) \hat{\varphi}(\omega) d\omega, \\ U(0, x) = \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{\varphi}(\omega) d\omega \end{cases} \quad (2.1)$$

at the same time, we have

$$\begin{cases} U_t(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} z_t(t, 0, \omega) \hat{\varphi}(\omega) d\omega, \\ U_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} i\omega z(t, 0, \omega) \hat{\varphi}(\omega) d\omega, \\ U_{x^2} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \omega^2 z(t, 0, \omega) \hat{\varphi}(\omega) d\omega, \\ U_{x^3} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} i\omega^3 z(t, 0, \omega) \hat{\varphi}(\omega) d\omega \end{cases} \quad (2.2)$$

Then from (1.1) it follows that:

$$\begin{cases} z_t(t, 0, \omega) - \lambda z(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \omega z(t, 0, \omega) \hat{\varphi}(\omega) d\omega \right)^2 - i\omega^3 z(t, 0, \omega) = (J\theta)(t), \\ z|_{t=0} = 1, \forall \omega \in R. \end{cases} \quad (2.3)$$

where z is a new unknown function.

Further, taking into account (1.3), (2.1), (2.2), and (2.3), we obtain

$$\begin{aligned} g(t) &= (U_t + U_{x,t})|_{x=0} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [z_t(t, 0, \omega) - i\omega^3 z(t, 0, \omega)] \hat{\varphi}(\omega) d\omega = \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\lambda z(t, 0, \omega) \times \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \omega z(t, 0, \omega) \hat{\varphi}(\omega) d\omega \right)^2 + \\ &+ i\omega^3 z(t, 0, \omega) + (J\theta)(t) - i\omega^3 z(t, 0, \omega)] \hat{\varphi}(\omega) d\omega = \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\lambda z(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \omega z(t, 0, \omega) \hat{\varphi}(\omega) d\omega \right)^2 + (J\theta)(t)] \hat{\varphi}(\omega) d\omega \end{aligned} \quad (2.4)$$

If

$$d = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(\omega) d\omega \neq 0, \quad (2.5)$$

then (2.4) implies

$$\begin{aligned} (J\theta)(t) &= d^{-1} \left\{ g(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\lambda z(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \omega z(t, 0, \omega) \hat{\varphi}(\omega) d\omega \right)^2 \hat{\varphi}(\omega) d\omega \right\} \equiv \\ &\equiv (B_\theta)(t). \end{aligned} \quad (2.6)$$

Therefore, substituting (2.6) into (2.3), we obtain

$$z_t(t, 0, \omega) = \lambda z(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \omega z(t, 0, \omega) \hat{\varphi}(\omega) d\omega \right)^2 + i\omega^3 z(t, 0, \omega) + (B_0 z)(t). \quad (2.7)$$

or,

$$\begin{aligned} z(t, 0, \omega) &= e^{i\omega^3 t} + \int_{-\infty}^t e^{i\omega^3(t-s)} \left[\lambda z(s, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \omega z(s, 0, \omega) \hat{\varphi}(\omega) d\omega \right)^2 + (B_0)(s) \right] ds \equiv \\ &\equiv (Bz)(t, 0, \omega). \end{aligned} \quad (2.8)$$

Since (2.8) is a Volterra integral equation of the second kind, we can formulate the following lemma:

Lemma 2.1. Under the conditions:

$$\begin{cases} L_s < 1, \\ B : S_r(z_0) \rightarrow S_r(z_0) = \{z : |z - z_0| \leq r = \text{const}, \forall (t, 0, \omega) \in \bar{D}\}, \end{cases} \quad (2.9)$$

the equation (2.8) is uniquely solvable in $C(\bar{D})$, and the solution is constructed by the Picard method:

$$Z_{n+1} = BZ_n, \quad (n = 0, 1, \dots),$$

with the estimate

$$\begin{cases} \|z_{n+1} - z\| \leq L_s^{n+1} r \xrightarrow{n \rightarrow \infty} 0, \\ |z| \leq r_1, \quad \forall (t, 0, \omega) \in \bar{D}, \end{cases} \quad (2.10)$$

where Z_0 is the initial approximation. Moreover, the function z has a derivative with respect to t satisfying the differential equation (2.7). Then, on the basis of (2.1), the function $U \in \tilde{W}^1(D)$ is uniquely determined in:

$$\|U\|_{W^1(D)} \leq (|d| + \sum_{i=0}^2 d_i) r_1 + N_0 = N_1. \quad (2.11)$$

3. Regularization of VIE-1 Degenerating on the Basis of (2.6)

Since the conditions of Lemma 2.1 hold, taking into account (1.4) and (2.6), we obtain the VIE-1:

$$J\theta \equiv \int_0^t K_0(t, s)\theta(s) ds + \lambda_1 \left(\int_0^t K_0(t, s)\theta(s) ds \right)^2 = F(t), \quad (3.1)$$

where:

$$C[0, T] \ni F(t) \equiv d^{-1} \left\{ g(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lambda \Psi(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \omega \Psi(t, 0, \omega) \hat{\varphi}(\omega) d\omega \right)^2 \hat{\varphi}(\omega) d\omega \right\}, \quad (3.2)$$

and conditions (b, c) are satisfied. In this case, equation (3.1) is an ill-posed VIE-1 under conditions (c) and (3.2). The regularization of VIE-1 (3.1) in the generalized sense in $Z^1(0, T)$, is proved on the basis of a modification of the method of singular perturbations, when conditions of the form:

$$\begin{cases} h(t) \equiv [\gamma + \frac{1}{\alpha}\mu(t)]F(t) \geq m > 0, (1 < \gamma = \text{const}); h_0(t) \equiv \gamma + \frac{1}{\alpha}\mu(t), \\ 0 \leq \mu(t) \in L^1(0, T); \phi(t) = \int_0^t [\gamma + \frac{1}{\alpha}\mu(s)]F(s)ds = \int_0^t h(s)ds, \\ |F_0(t) - F_0(s)| \leq C_1(\phi(t) - \phi(s)), \quad (s \leq t; \gamma > 1; M_0 = \frac{1}{\gamma\alpha}; C_1 = L_{F_0}M_0), \\ t - s \leq M_0(\phi(t) - \phi(s)), \quad (s \leq t), t \in [0, T]: \quad t = (t^{\frac{7}{2}})^{\frac{7}{2}} \leq (\phi(t))^{\frac{7}{2}}, \\ \left(\mu(t) = \frac{2}{\sqrt[7]{t^5}} \right), t \leq M_1(\phi(t))^2, \quad \left(M_1 = \sup_{[0, T]} (\phi(t))^{\frac{3}{2}} \right). \end{cases} \quad (3.3)$$

are admissible permissible. For this purpose, transforming (3.1) to the form

$$\begin{cases} \int_0^t h(s)\Theta(s) ds = (H\Theta)(t) + F(t), \\ (H\Theta)(t) \equiv \int_0^t h_0(s)\Theta(s)(J\Theta)(s) ds - (J\Theta)(t) = \\ = \int_0^t h_0(s)\Theta(s) \left\{ \int_0^s K_0(s, \bar{s})\Theta(\bar{s}) d\bar{s} + \lambda_1 \left(\int_0^s K_0(s, \bar{s})\Theta(\bar{s}) d\bar{s} \right)^2 \right\} ds - \\ - \int_0^t K_0(t, \bar{s})\Theta(\bar{s}) d\bar{s} - \lambda_1 \left(\int_0^t K_0(t, \bar{s})\Theta(\bar{s}) d\bar{s} \right)^2. \end{cases} \quad (3.4)$$

We introduce a parameterized system of integral equations:

$$\begin{cases} \varepsilon\Theta_\varepsilon(t) + (\Phi\Theta_\varepsilon)(t) = F_\varepsilon(t), \\ (\Phi\Theta_\varepsilon)(t) = \int_0^t h(s)\Theta_\varepsilon(s) ds - (H\Theta_\varepsilon)(t). \end{cases} \quad (3.5)$$

Then, representing the solutions according to the rule

$$\begin{cases} \theta_\varepsilon = v(t) + \xi_\varepsilon(t) + \frac{1}{\varepsilon}\Pi_\varepsilon(t), \quad (\Pi_\varepsilon(0) = F(0), v(0) = 0, \xi_\varepsilon(0) = 0), \\ F_\varepsilon(0) = F(0); F_\varepsilon(t) : \\ |F_\varepsilon(t) - F(t)| \leq \Delta_0(\varepsilon), \quad \forall t \in [0, T], \end{cases} \quad (3.6)$$

we obtain the system:

$$\begin{cases} \varepsilon\Pi_\varepsilon(t) + \int_0^t h(s)\Pi_\varepsilon(s) ds = F(0), \\ \int_0^t h(s)v(s) ds = (Hv)(t) + F_0(t), \quad (F_0(t) \equiv F(t) - F(0)), \\ \varepsilon\xi_\varepsilon(t) + \int_0^t h(s)\xi_\varepsilon(s) ds = (H[v + \xi_\varepsilon + \frac{1}{\varepsilon}\Pi_\varepsilon])(t) - (Hv)(t) + F_\varepsilon(t) - F(t) - \varepsilon v(t). \end{cases} \quad (3.7)$$

Lemma 3.1. If conditions (b), (c), and (3.2) are satisfied, then from system (3.7) the following statements hold:

$$1) |\Pi_\varepsilon(t)| \leq C_0 \exp\left(-\frac{1}{\varepsilon}\varphi(t)\right), \quad (3.8)$$

2) the function $v(t) \in C[0, T]$ is defined as the limit of the parameterized integral equation

$$\delta v(s) + \int_0^s h(s)v(s) ds = (Hv_\delta)(t) + F_0(t) \quad (3.9)$$

in $C[0, T]$, where $(0, 1) \ni \delta$ is a small parameter,

3) the residual function $\xi_\varepsilon(t)$ is uniquely determined from the third equation of system (3.7) and converges uniformly to zero in $C[0, T]$, as $\varepsilon \rightarrow 0$.

Theorem 3.1. If the conditions of Lemma 3.1. hold, then on the basis of (3.6) the following statements are valid:

1)

$$\begin{cases} \|\Pi_\varepsilon\|_{Z^1(0, T)} \leq \gamma_1 \varepsilon^2, \quad (\gamma_1 = 6C_0M_1), \\ \|\Omega_\varepsilon\|_{Z^1(0, T)} \leq \gamma_1 \varepsilon, \end{cases} \quad (3.10)$$

2)

$$\begin{cases} \|\theta_\varepsilon - v\|_{L^2(0, T)} \leq E_1(\varepsilon)T + \gamma_1 \varepsilon = \tilde{M}_0(\varepsilon), \\ \|\theta_\varepsilon\|_{L^2(0, T)} \leq r_* = \text{const}, \end{cases} \quad (3.11)$$

3)

$$\|(\Phi\theta_\varepsilon)(t) - F(t)\|_{L^2(0, T)} \leq \tilde{M}(\varepsilon), \quad (\tilde{M}_0(\varepsilon), \tilde{M}_0(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0). \quad (3.12)$$

Proof. 1) Indeed, condition (3.10) follows from (3.8) of Lemma 3.1. For this purpose, first integrating the inequality (3.8) with respect to the variable t , we obtain

$$\begin{cases} \int_0^T |\Pi_\varepsilon(s)| ds \leq C_0 \int_0^T \exp\left(-\frac{1}{\varepsilon}\phi(s)\right) ds = C_0 \left[t \exp\left(-\frac{1}{\varepsilon}\phi(t)\right) + \int_0^t s \exp\left(-\frac{1}{\varepsilon}\phi(s)\right) d\left(\frac{1}{\varepsilon}\phi(s)\right) \right] \leq \\ \leq C_0 M_1 \varepsilon^2 \left[\left(\frac{1}{\varepsilon}\phi(t)\right)^2 \exp\left(-\frac{1}{\varepsilon}\phi(t)\right) + \int_0^t \frac{1}{\varepsilon}\phi(s) \rho^2 \exp\left(-\frac{1}{\varepsilon}\phi(s)\right) d\left(\frac{1}{\varepsilon}\phi(s)\right) \right] \leq \\ \leq C_0 M_1 \varepsilon^2 \left[4e^{-2} + \int_0^\infty \rho^2 e^{-\rho} d\rho \right] = 6C_0 M_1 \varepsilon^2 = \gamma_1 \varepsilon^2, \end{cases}$$

or, taking into account the norm $Z^1(0, T)$ we obtain (3.10):

$$\begin{cases} \|\Pi_\varepsilon\|_{Z^1(0, T)} \leq \gamma_1 \varepsilon^2, \\ \|\Omega_\varepsilon\|_{Z^1(0, T)} \leq \gamma_1 \varepsilon. \end{cases}$$

2) Next, we proceed to the proof of (3.11). Taking into account (3.6) and performing an estimate, we obtain:

$$\begin{cases} |\theta_\varepsilon - v| \leq E_1(\varepsilon) + \frac{1}{\varepsilon} |\Pi_\varepsilon(t)| \leq E_1(\varepsilon) + \frac{1}{\varepsilon} C_0 \exp\left(-\frac{1}{\varepsilon}\phi(t)\right), \\ \|\Pi_\varepsilon\| \leq E_1(\varepsilon) + C_0 \exp\left(-\frac{1}{\varepsilon}\phi(t)\right) + \tilde{r}_2. \end{cases} \quad (3.13)$$

On the other hand, estimating (3.13) in the sense of the norm $Z^1(0, T)$, we have:

$$\|\theta_\varepsilon - v\|_{Z^1} \leq E_1(\varepsilon)T + \gamma_1 \varepsilon = \tilde{M}_0(\varepsilon),$$

where

$$\|\theta_\varepsilon\|_{Z^1(0, T)} \leq T(\tilde{r}_2 + E_1(\varepsilon)) + \gamma_1 \varepsilon \leq r.$$

Hence, relation (3.11) holds.

3) To prove (3.12), we first take into account the estimate:

$$\|(\Phi\theta_\varepsilon)(t) - F(t)\| = |\varepsilon\theta_\varepsilon + (\Phi\theta_\varepsilon)(t) - F_\varepsilon(t) - \varepsilon(\theta_\varepsilon - v + v) + F_\varepsilon(t) - F(t)|, \quad (3.14)$$

where $F(t)$ is the free term of VIE-1 (3.1) (or (3.4)), and the operator $(\Phi\theta_\varepsilon)(t)$ is defined as in (3.5). Further, estimating in $Z^1(0, T)$, we obtain

$$\begin{aligned} \|(\Phi\theta_\varepsilon)(t) - F(t)\|_{Z^1(0, T)} &\leq \|F_\varepsilon(t) - F(t)\|_{Z^1} + \varepsilon\|\theta_\varepsilon(t) - v(t)\|_{Z^1} + \varepsilon\tilde{r}_2 T \leq \\ &\leq \Delta_0(\varepsilon)T + \varepsilon\tilde{M}_0(\varepsilon) + \varepsilon\tilde{r}_2 T = \tilde{M}(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

which was to be proved.

From the obtained results of Lemma 2.1. and Theorem 3.1., on the basis of (2.1), (2.3),

and the function $z, z_t(t, 0, \omega)$, and taking into account (3.4), we finally obtain:

$$\left\{ \begin{array}{l} \int_0^t h(s)\theta(s) ds - (H\theta)(t) \equiv (J\theta)(t) = F(t), \\ H\theta \equiv \int_0^t h_0(\tau)\theta(\tau)(J\theta)(\tau) d\tau - (J\theta)(t), \quad \int_0^t h(s)\theta(s) ds \equiv \int_0^t h_0(s)\theta(s)(J\theta)(s) ds, \\ U(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} ([e^{i\omega^3 t} + \int_0^t e^{i\omega^3(t-s)} (\lambda z(s, 0, \omega) (\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z(s, 0, \bar{\omega}) \hat{\phi}(\bar{\omega}) d\bar{\omega})^2 + \\ + \int_0^s h(s')\theta(s') ds' - (H\theta)(s))] \hat{\phi}(\omega) d\omega. \\ U_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} i\omega ([e^{i\omega^3 t} + \int_0^t e^{i\omega^3(t-s)} (\lambda z(s, 0, \omega) (\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z(s, 0, \bar{\omega}) \hat{\phi}(\bar{\omega}) d\bar{\omega})^2 + \\ + \int_0^s h(s')\theta(s') ds' - (H\theta)(s))] \hat{\phi}(\omega) d\omega. \\ U_{x^2} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} i\omega^2 ([e^{i\omega^3 t} + \int_0^t e^{i\omega^3(t-s)} (\lambda z(s, 0, \omega) (\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z(s, 0, \bar{\omega}) \hat{\phi}(\bar{\omega}) d\bar{\omega})^2 + \\ + \int_0^s h(s')\theta(s') ds' - (H\theta)(s))] \hat{\phi}(\omega) d\omega. \\ U_{x^3} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} i\omega^3 ([e^{i\omega^3 t} + \int_0^t e^{i\omega^3(t-s)} (\lambda z(s, 0, \omega) (\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z(s, 0, \bar{\omega}) \hat{\phi}(\bar{\omega}) d\bar{\omega})^2 + \\ + \int_0^s h(s')\theta(s') ds' - (H\theta)(s))] \hat{\phi}(\omega) d\omega. \\ U_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} (\lambda z(s, 0, \omega) (\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z(s, 0, \bar{\omega}) \hat{\phi}(\bar{\omega}) d\bar{\omega})^2 + \\ + i\omega^3 z(t, 0, \omega) + \int_0^s h(s')\theta(s') ds' - (H\theta)(s))] \hat{\phi}(\omega) d\omega. \end{array} \right. \quad (3.15)$$

From this it follows that, when carrying out estimates for the expressions

$$U - U_\varepsilon; \quad U_x - U_{x\varepsilon}; \quad U_{x^2} - U_{x^2\varepsilon}; \quad U_{x^3} - U_{x^3\varepsilon}; \quad U_t - U_{t\varepsilon}$$

there appears the expression: $U_t - U_{t\varepsilon}$, associated with the difference:

$$\left\{ \begin{array}{l} (\Phi\theta_\varepsilon)(t) - F(t), \\ (\Phi\theta_\varepsilon)(t) = \int_0^t h(s)\theta_\varepsilon(s) ds - (H\theta_\varepsilon)(t), \\ F(t) = \int_0^t h(s)\theta(s) ds - (H\theta)(t). \end{array} \right. \quad (3.16)$$

where the difference (3.16) is not under the integral sign with respect to the variable $t \in [0, T]$, which distinguishes it from other expressions.

Therefore, we perform an estimate with respect to the expression:

$$U_{t\varepsilon} - U_t$$

where this expression is constructed by taking (3.15) into account, i.e.,

$$\left\{ \begin{aligned} U_{t\varepsilon} - U_t &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \left\{ \lambda[z_\varepsilon(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z_\varepsilon(t, 0, \bar{\omega}) \hat{\varphi}(\bar{\omega}) d\bar{\omega} \right)^2 - \right. \\ &\quad \left. - z(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z(t, 0, \bar{\omega}) \hat{\varphi}(\bar{\omega}) d\bar{\omega} \right)^2 + i\omega^3 [z_\varepsilon(t, 0, \omega) - z(t, 0, \omega)] + \right. \\ &\quad \left. + (\Phi\theta_\varepsilon)(t) - F(t) \right\} \hat{\varphi}(\omega) d\omega. \end{aligned} \right. \quad (3.17)$$

Then, on the basis of the results of Lemma 2.1. and estimate (2.10) for the functions Z, Z_ε , assuming the conditions:

$$\left\{ \begin{aligned} |z_\varepsilon - z| &\leq \Delta_{01}(\varepsilon), \quad \forall (t, 0, \omega) \in \bar{D}, \\ |z|, |z_\varepsilon| &\leq r_1, \quad \forall (t, 0, \omega) \in \bar{D}; \quad \Delta_{01}(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0, \end{aligned} \right. \quad (3.18)$$

it follows from estimate (3.17) that:

$$\begin{aligned} |U_{t\varepsilon} - U_t| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \left\{ \lambda[z_\varepsilon(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z_\varepsilon(t, 0, \bar{\omega}) \hat{\varphi}(\bar{\omega}) d\bar{\omega} \right)^2 - \right. \right. \\ &\quad \left. \left. - z(t, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z(t, 0, \bar{\omega}) \hat{\varphi}(\bar{\omega}) d\bar{\omega} \right)^2 \right\} + \right. \\ &\quad \left. + i\omega^3 [z_\varepsilon(t, 0, \omega) - z(t, 0, \omega)] + (\Phi\theta_\varepsilon)(t) - F(t) \right\} \hat{\varphi}(\omega) d\omega \leq \\ &\leq |\lambda| \times |d| (3r_1 d_1 + d_2) \Delta_{01}(\varepsilon) + |d| |(\Phi\theta_\varepsilon)(t) - F(t)| = \\ &= d_* \Delta_{01}(\varepsilon) + |d| |(\Phi\theta_\varepsilon)(t) - F(t)|, \quad (d_* = |\lambda| \times |d| (3r_1 d_1 + d_2)). \end{aligned}$$

Consequently, integrating with respect to t , we obtain:

$$\int_0^t |U_{t\varepsilon}(s, x) - U_t(s, x)| ds \leq d_* \Delta_{01}(\varepsilon) T + |d| \int_0^t |(\Phi\theta_\varepsilon)(s) - F(s)| ds \leq d_* \Delta_{01}(\varepsilon) T + \tilde{M}(\varepsilon).$$

Therefore, we have

$$\|U_{t\varepsilon} - U_t\|_1 \leq d_* \Delta_{01}(\varepsilon) T + \tilde{M}(\varepsilon) = Y_0(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.19)$$

$U_\varepsilon - U$.

From the remaining expressions, for clarity we provide an estimate for

Note that this and the other expressions:

$$U_x - U_{x\varepsilon}; \quad U_{x^2} - U_{x^2\varepsilon}; \quad U_{x^3} - U_{x^3\varepsilon},$$

contain (3.16) under the time integral sign; therefore, their estimates are analogous to each other.

Indeed, from the equality

$$\begin{aligned} U_\varepsilon - U &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \left[\int_0^t e^{i\omega s} \left\{ \lambda(z_\varepsilon(s, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z_\varepsilon(s, 0, \bar{\omega}) \hat{\varphi}(\bar{\omega}) d\bar{\omega} \right)^2 - \right. \right. \\ &\quad \left. \left. - z(s, 0, \omega) \left(\frac{1}{\sqrt{2\pi}} \int_R \bar{\omega} z(s, 0, \bar{\omega}) \hat{\varphi}(\bar{\omega}) d\bar{\omega} \right)^2 \right\} + (\Phi\theta_\varepsilon)(s) - F(s) \right] \hat{\varphi}(\omega) d\omega, \end{aligned}$$

the estimate follows:

$$\begin{aligned} \|U_\varepsilon - U\|_{C(D)} &\leq 3|\lambda| \times |d|r_1 d_1 T \Delta_{01}(\varepsilon) + |d| \int_0^t |(\Phi\theta_\varepsilon)(s) - F(s)| ds \leq \\ &\leq |d|[3|\lambda|r_1 d_1 T \Delta_{01}(\varepsilon) + \tilde{M}(\varepsilon)] = Y_1(\varepsilon), \quad \forall(t, x) \in D \end{aligned}$$

or

$$\|U_\varepsilon - U\|_{C(D)} \leq \Upsilon_1(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.20)$$

Therefore, as noted above, analogously we may admit the estimates

$$\begin{cases} \|U_{x\varepsilon} - U_x\|_{C(D)} \leq \Upsilon_2(\varepsilon), \\ \|U_{x^2\varepsilon} - U_{x^2}\|_{C(D)} \leq \Upsilon_3(\varepsilon), \\ \|U_{x^3\varepsilon} - U_{x^3}\|_{C(D)} \leq \Upsilon_4(\varepsilon). \end{cases} \quad (3.21)$$

Hence, taking into account (3.19) and (3.21), we have:

$$\|U_\varepsilon - U\|_{\tilde{W}^1(D)} \leq \sum_{i=0}^4 \Upsilon_i(\varepsilon) = \Upsilon_*(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.22)$$

As a result, taking into account (3.11) and (3.22) together with the conclusions stated above, we obtain

$$\|V_\varepsilon\|_{\tilde{G}_{[\tilde{W}(D); Z^1(0,T)]}^{1,1}(D)} \leq \Upsilon_*(\varepsilon) + \tilde{M}_0(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.23)$$

Statement 3.1. Under the conditions of Lemma 2.1., Theorem 3.1., and (3.23), the IP (1.1), (1.2), and (1.3) is regularizable in the generalized sense in $\tilde{G}_{[\tilde{W}(D); Z^1(0,T)]}^{1,1}(D)$.

If, with respect to (1.4), we assume: $\lambda_1 = 0$, then instead of the integral equation (3.1) we obtain the linear ill-posed VIE-1:

$$\int_0^t K_0(t, s)\theta(s) ds = F(t), \quad (3.24)$$

and all results of Theorem 3.1. remain valid for (3.24).

Therefore, for the IP in an unbounded domain, where a VIE-1 of the form (3.24) degenerates, we can draw analogous conclusions to those stated above. Hence, in the final analysis, the IP degenerating into VIE-1 (3.24) satisfies the conditions of Statement 1.

4. Conclusion

In this paper, we consider an inverse problem for a loaded Korteweg-de Vries type operator in an unbounded domain, which degenerates into various types of Volterra integral equations of the first kind.

The study is based on the use of the FT apparatus and RM in specially introduced functional spaces. In particular, the case of an ill-posed nonlinear VIE-1 of a special form is analyzed, in which the question of regularizability is resolved in the generalized sense.

5. Acknowledgments

The authors thank the reviewers for their valuable comments that helped improve the quality of this manuscript.

References

- [1] T.M. Imanaliev, *Justification and development of the method of an additional argument for solving partial differential equations*, -Bishkek: Doctoral dissertation (Dr. Sci. in Phys.-Math.), Specialty 01.01.02, Central Scientific Library of the NAS of the Kyrgyz Republic, (2000).
- [2] M.I. Imanaliev, P.S. Pankov, T.M. Imanaliev, *The method of an additional argument in the theory of nonlinear wave partial differential equations*, Doklady Akademii Nauk (DAN), **343**:5 (1995), 546–548.
- [3] A. Newell, *Solitons in Mathematics and Physics*, -Moscow: Mir (Russian translation), (1989).
- [4] G. Whitham, *Linear and Nonlinear Waves*, Moscow: Mir, (1977).
- [5] L.D. Faddeev, *Poisson structure for the KdV equation*, Lett. Math. Phys., **10** (1985), 183–188.
- [6] R.M. Miura, *Korteweg - de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Math. Phys., **9** (1968), 1202–1204.
- [7] V.A. Trenogin, *Functional Analysis*, Moscow: Nauka, (1980).
- [8] A.N. Tychonoff, V.Ya. Arsenin, *Methods for Solving Ill-Posed Problems*, Moscow: Nauka, (1974).