

Singularly Perturbed Equations with Analytic Functions

K.S. Alybaev^{*1}, M. Nurmatova²

^{1,2} *Jalal-Abad State University named after B. Osmonov, Manas, Kyrgyzstan*

Abstract

In this work, we investigate the asymptotic behavior of solutions of first-order singularly perturbed equations with analytic functions in the general case. To study the asymptotics, methods of conformal mapping, level lines, and other asymptotic techniques are employed. The existence of boundary-layer lines and regions where limiting transitions occur is proved. A distinctive feature of the considered case is that the analysis is carried out without invoking the stability of equilibrium points or stationary points.

Keywords: singularly perturbed equations, equilibrium states, instability delay, autonomous and non-autonomous equations, analytic functions.

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1. Introduction

Differential equations with small parameters in the highest derivatives are called singularly perturbed equations.

Such equations arise in the study of various processes in physics, mechanics, oscillation theory, automatic and optimal control, biology, chemistry, medicine, and other scientific fields. Here, we focus only on works related to the study of singularly perturbed ordinary differential equations [1, 2, 3].

Consider the equations

$$\varepsilon x' = f(x, y), \quad (1.1)$$

$$y' = g(x, y), \quad (1.2)$$

$$\varepsilon x' = f(t, x, y), \quad (1.3)$$

$$y' = g(t, x, y), \quad (1.4)$$

where $0 < \varepsilon$ is a small real parameter, $t \in [t_0, T]$, and $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$.

Equations (1.1)–(1.2) and (1.3)–(1.4) are called autonomous and non-autonomous, respectively. The study of autonomous singularly perturbed equations is discussed in [1], while non-autonomous cases are considered in [2].

One of the central problems in the theory of singularly perturbed equations is the limiting transition problem.

The problem is as follows. Setting $\varepsilon = 0$ in (1.1)–(1.2) and (1.3)–(1.4) yields the so-called degenerate equations:

$$0 = f(x, y), \quad (1.5)$$

$$y' = g(x, y), \quad (1.6)$$

$$0 = f(t, x, y), \quad (1.7)$$

$$y' = g(t, x, y). \quad (1.8)$$

^{*}Corresponding author: alybaevkurmanbek@rambler.ru

If the degenerate equations admit solutions $x = \varphi(t), y = \psi(t), t \in [t_0, T]$, then the following question arises.

Let $x(t, \varepsilon)$ and $y(t, \varepsilon)$ be solutions to systems (1.1)–(1.2), (1.3)–(1.4) satisfying the initial conditions

$$x(t_0, \varepsilon) = x^0, \quad y(t_0, \varepsilon) = y^0 \quad (1.9)$$

Problem. Under what conditions does the relation

$$\lim_{\varepsilon \rightarrow 0} x(t, \varepsilon) = \varphi(t), \quad \lim_{\varepsilon \rightarrow 0} y(t, \varepsilon) = \psi(t). \quad (1.10)$$

hold? The problem in the most general form for autonomous equations is solved in [1], and for non-autonomous equations in [2, 3].

If the right-hand sides of (1.1)–(1.2) or (1.3)–(1.4) are analytic in all variables in some domain, such equations are called singularly perturbed equations with analytic functions.

Some classes of singularly perturbed equations with analytic functions in certain regions of the complex plane of the independent variable have been considered in [4, 5, 6, 7, 8, 9]. New concepts such as boundary-layer line, boundary-layer region, singular and regular regions, and attraction region have been introduced, and their existence is proved. These studies are carried out without invoking equilibrium point stability [1, 2, 3]. In this work, we consider a first-order singularly perturbed equation with analytic functions and solve the limiting transition problem, proving the existence of boundary layers.

2. Main Results

Consider the equation

$$\varepsilon x' = f(t, x) \quad (2.1)$$

with initial condition

$$x(t_0, \varepsilon) = x^0, \quad (2.2)$$

where $0 < \varepsilon$ is a small real parameter, and $t \in \mathcal{D} \subset \mathbb{C}$ is a complex domain.

U1. $f(t, x) \in Q(H)$ the space of analytic functions in $H = \{(t, x) : t \in \mathcal{D}, |x - x^0| < C_1\}$.

Here, $C_1, C_2, \dots$ denote positive constants independent of ε . From (2.1), the degenerate equation is:

$$f(t, \xi) = 0. \quad (2.3)$$

U2. Let $\xi = \varphi(t)$ be a solution of (2.3) with $\varphi(t) \in Q(\mathcal{D})$.

We investigate whether there exists a region where

$$\lim_{\varepsilon \rightarrow 0} x(t, \varepsilon) = \varphi(t).$$

Equation (2.1) can be expanded as

$$\varepsilon x' = f'_x(t, \varphi(t))(x - \varphi(t)) + \frac{1}{2!} f''_{x^2}(t, \varphi(t))(x - \varphi(t))^2 + \dots \quad (2.4)$$

Introducing $z = x - \varphi(t)$ and denoting:

$$f'_x(t, \varphi(t)) \equiv a(t), \frac{1}{2!} f''_{x^2}(t, \varphi(t)) z^2 + \dots \equiv g(t, z)$$

(2.4) becomes

$$\varepsilon z' = a(t)z + \varepsilon b(t) + g(t, z) \quad (2.5)$$

with

$$z(t_0, \varepsilon) = x^0 - \varphi(t_0), \quad (2.6)$$

$b(t) \equiv -\varphi'(t)$. By **U1–U2**:

$$a(t) \in Q(\mathcal{D}), \quad b(t) \in Q(\mathcal{D}), \quad g(t, z) \in Q(H),$$

$$H = \{(t, z) : t \in \mathcal{D}, |z| < C_1\}.$$

U3. $\forall t \in \mathcal{D} \ (a(t) \neq 0)$.

U4. For all $\forall((t, \tilde{z}), (t, \tilde{\tilde{z}})) \in H : |g(t, \tilde{z}) - g(t, \tilde{\tilde{z}})| \leq C_2 |\tilde{z} - \tilde{\tilde{z}}| \max\{|\tilde{z}|, |\tilde{\tilde{z}}|\}$.

3. Asymptotic Representation

The solution of (2.5)–(2.6) can be represented as.

$$z = z^0 \exp\left(\frac{A(t)}{\varepsilon}\right) + \frac{1}{\varepsilon} \int_{t_0}^t [\varepsilon b(\tau) + g(\tau, z)] \exp\left(\frac{A(t) - A(\tau)}{\varepsilon}\right) d\tau, \quad (3.1)$$

where $A(t) = \int_{t_0}^t a(\tau) d\tau$.

By **U3**, $A(t)$ has a simple zero at t_0 and no multiple zeros in $\mathcal{D}$.

4. Level Lines Method

$$(p) = \{t \in \mathcal{D} : \operatorname{Re} A(t) = p = \text{const}\},$$

$$(q) = \{t \in \mathcal{D} : \operatorname{Im} A(t) = q = \text{const}\}.$$

Level lines passing through t_0 are denoted

$$(p_0) = \{t \in \mathcal{D} : \operatorname{Re} A(t) = 0\},$$

$$(q_0) = \{t \in \mathcal{D} : \operatorname{Im} A(t) = 0\}.$$

level lines (p) and (q) intersect orthogonal [10]. The line (p_0) divides $\mathcal{D}$ into regions $\mathcal{D}_1$ and $\mathcal{D}_2$. Pick an arbitrary point $\tilde{t}$ on (p_0) and draw the line $(\tilde{q})$. Along $(\tilde{q})$. $\operatorname{Re} A(t)$ is strictly monotonic. Since $\operatorname{Re} A(\tilde{t}) = 0$, one of the following holds

$\operatorname{Re} A(t) \leq 0, \forall t \in \mathcal{D}_1$, or $\operatorname{Re} A(t) \geq 0, \forall t \in \mathcal{D}_2$, with equality only on (p_0) . Assume $\operatorname{Re} A(t) \leq 0, \forall t \in \mathcal{D}_1$ implies $\operatorname{Re} A(t) \geq 0, \forall t \in \mathcal{D}_2$.

On the line (p_0) , select points T_{11}, T_{12} on both sides of the point t_0 and draw the level lines $(q_1), (q_2)$.

Similarly, on the line (q_0) , select points T_{21}, T_{22} and draw lines $(p_1), (p_2)$ (Fig. 1).

Without loss of generality, assume that the lines $(p_1), (p_2), (q_1), (q_2)$ form a curvilinear quadrilateral $\mathcal{D}_0 \subset \mathcal{D}$.

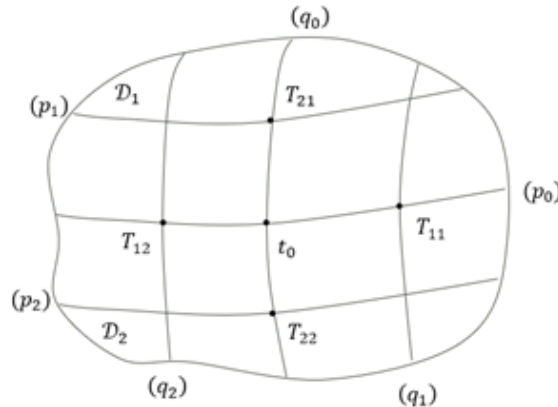


Figure 1: Lines $(p_1), (p_2), (q_1), (q_2)$

5. Conformal Mapping

For simplification of further analysis in (3.1), introduce the conformal transformation

$$A(t) = u \quad (5.1)$$

By **U3**, this transformation is bijective and conformal. It maps the domain $\mathcal{D}_0$ to a rectangle Π_0 , centered at the origin $(0; 0)$ in the u -plane (Fig. 2).

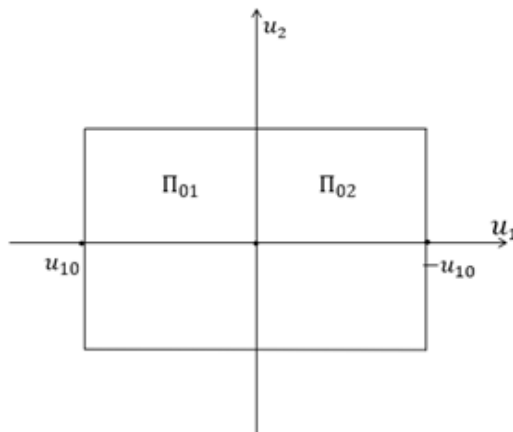


Figure 2: Rectangle Π_0 .

The line (p_0) maps to the axis u_2 , and (q_0) maps to the axis u_1 . Denote:

$$\mathcal{D}_{01} \subset \mathcal{D}_0 \wedge \forall t \in \mathcal{D}_{01} \quad (\operatorname{Re} A(t) \leq 0),$$

$$\mathcal{D}_{02} \subset \mathcal{D}_0 \wedge \forall t \in \mathcal{D}_{02} \quad (\operatorname{Re} A(t) \geq 0),$$

which map to Π_{01} and Π_{02} respectively.

Then equation (3.1) becomes:

$$\tilde{z} = z^0 \exp \frac{u}{\varepsilon} + \frac{1}{\varepsilon} \int_0^u \left[\varepsilon \tilde{b}(\tilde{u}) + g(\tilde{u}, \tilde{z}) \right] \exp \frac{u - \tilde{u}}{\varepsilon} d\tilde{u}, \quad (5.2)$$

where

$$\begin{aligned} \tilde{z} &\equiv z(A^{-1}(\tilde{u}), \varepsilon), \quad \tilde{b}(\tilde{u}) = b(A^{-1}(\tilde{u}), (A^{-1}(\tilde{u}))'), \\ g(\tilde{u}, \tilde{z}) &\equiv g(A^{-1}(\tilde{u}), z(A^{-1}(\tilde{u}), \varepsilon), (A^{-1}(\tilde{u}))'), \text{ where} \\ u &= u_1 + iu_2, \quad \tilde{u} = \tilde{u}_1 + i\tilde{u}_2. \end{aligned}$$

Then equation (5.1) can be considered along the u_1 axis contained in Π_{01} giving.

$$\tilde{z} = z^0 \exp \frac{u_1}{\varepsilon} + \frac{1}{\varepsilon} \int_0^{u_1} \left[\varepsilon \tilde{b}(\tilde{u}_1) + g(\tilde{u}_1, \tilde{z}) \right] \exp \frac{u_1 - \tilde{u}_1}{\varepsilon} d\tilde{u}_1, \quad (5.3)$$

and applying the results of [2, 3], we obtain the estimate:

$$|\tilde{z}| \leq C_3 \begin{cases} C_0, & \varepsilon \ln \varepsilon < u_1 \leq 0; \\ \varepsilon, & u_{10} \leq u_1 \leq \varepsilon \ln \varepsilon. \end{cases} \quad (5.4)$$

The interval $(\varepsilon \ln \varepsilon, 0]$ corresponds to the boundary layer. According to estimate (20), at $u_1 = \varepsilon \ln \varepsilon$, the solution (5.1) is of order $O(\varepsilon)$, and corresponds to the boundary layer. According to estimate $u_{10} \leq u_1 \leq \varepsilon \ln \varepsilon$.

We divide the line $u_1 = \varepsilon \ln \varepsilon$ in Π_{01} into the subdomains $\Pi_{0,-\varepsilon}$ and Π_{03} , and the line $u_1 = -\varepsilon \ln \varepsilon$ in Π_{02} into $\Pi_{0,\varepsilon}$ and Π_{04} (Fig. 3).

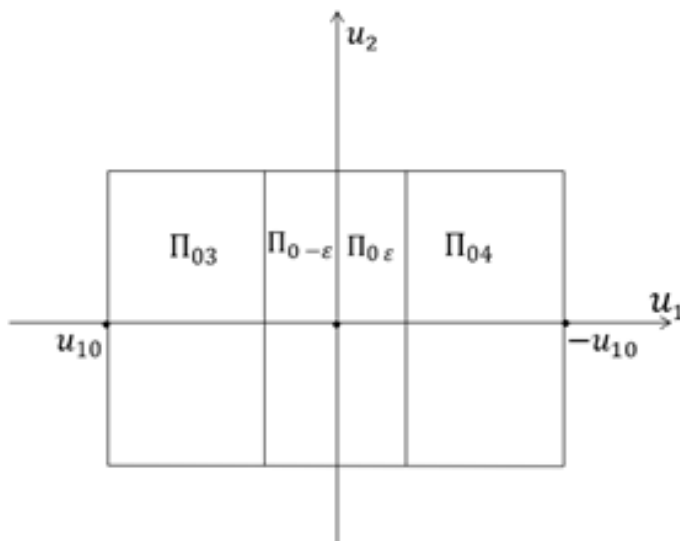


Figure 3: Regions $\Pi_{0,-\varepsilon}, \Pi_{0,\varepsilon}, \Pi_{03}, \Pi_{04}$.

Equation (5.1) can be further rewritten in the form

$$\begin{aligned} \tilde{z} = \exp \frac{i u_2}{\varepsilon} \left[z^0 \exp \frac{u_1}{\varepsilon} + \frac{1}{\varepsilon} \int_0^{u_1} \left[\varepsilon \tilde{b}(\tilde{u}_1) + g(\tilde{u}_1, \tilde{z}) \right] \exp \frac{u_1 - \tilde{u}_1}{\varepsilon} d\tilde{u}_1 \right] + \\ + \frac{1}{\varepsilon} \int_0^{u_2} \left[\varepsilon \tilde{b}(i \tilde{u}_2) + g(i \tilde{u}_2, \tilde{z}) \right] \exp \frac{i u_2 - i \tilde{u}_2}{\varepsilon} i d\tilde{u}_2 \end{aligned} \quad (5.5)$$

which provides the solution of (5.1) for $u_{10} \leq u_1 \leq 0$. In $u \in \Pi_{03}$ ($u_{10} \leq u_1 \leq \varepsilon \ln \varepsilon$) this solution remains of order $O(\varepsilon)$.

Accordingly, equation (5.5) can be expressed as

$$\tilde{z} = \tilde{z}(u_1, \varepsilon) \exp \frac{i u_2}{\varepsilon} + \frac{1}{\varepsilon} \int_0^{u_2} \left[\varepsilon \tilde{b}(i \tilde{u}_2) + g(i \tilde{u}_2, \tilde{z}) \right] \exp \frac{i u_2 - i \tilde{u}_2}{\varepsilon} i d\tilde{u}_2 \quad (5.6)$$

where $|\tilde{z}(u_1, \varepsilon)| \leq C_3 \varepsilon$ for $u \in \Pi_{03} \subset \Pi_{01}$.

Applying the method of successive approximations to (5.6), we obtain the estimate:

$$|\tilde{z}| \leq C_4 \varepsilon, \quad u \in \Pi_{03} \subset \Pi_{01} \quad (5.7)$$

6. Conclusion

Hence, due to the one-to-one conformal transformation, there exists a domain $\mathcal{D}_{00} \subset \mathcal{D}_1$ such that:

$$\forall t \in \mathcal{D}_{00} (|x(t, \varepsilon) - \varphi(t)| \leq C_4 \varepsilon).$$

Hence, the solution of the posed problem follows.

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References

- [1] L.S. Pontryagin, E.F. Mishchenko, *Some questions of the theory of differential equations with small parameters*, Proc. Steklov Inst. Math. **169** (1985), 99–188.
- [2] A.N. Tychonoff, *Systems of differential equations containing small parameters in derivatives*, Mathematical Collection. **31**:73 (1952), 575–586.
- [3] A.B. Vasilyeva, V.F. Butuzov, *Asymptotic methods in the theory of singular perturbations*, — Moscow: Higher School, (1990).
- [4] K.S. Alybaev, *Level line method for studying singularly perturbed equations with stability condition violation*, Bulletin of KSU. Series, **3**:6 (2001), 190–200.
- [5] P.S. Pankov, K.S. Alybaev, M.R. Narbaev, K.B. Tampagarov, *The phenomenon of boundary layer lines and asymptotics of solutions of singularly perturbed linear ordinary differential equations with analytic functions*, Bulletin of Osh State University, **1** (2013), 227–231.
- [6] K.S. Alybaev, A.B. Murzabaeva, *Singularly perturbed first-order equations in complex domains that lose their uniqueness under degeneracy. In “International Conference on Analysis and Applied Mathematics*, AIP Conference Proceedings, **1997** (2018).
- [7] K.S. Alybaev, T.K. Narymbetov, *Regions of attraction of solutions of singularly perturbed systems of equations*, Vestnik JASU. **1** (2021), 5–8.
- [8] K.S. Alybaev, A.M. Dzhuraev, M.N. Nurmatova, *Delay in solving autonomous singularly perturbed equations near an unstable equilibrium position*, Lobachevskii Journal of Mathematics, **45**:3 (2024), 912–921.
- [9] M.N. Nurmatova, *Influence of turning points on the delay in solving near an unstable equilibrium position*, Alatau Academic Studies, **2** (2024), 398–411.
- [10] M.A. Lavrentiev, B.V. Shabat, *Methods of the theory of complex variable functions*, — Moscow: Nauka, (1973).