

On Homeomorphism Groups and the Set-Open Topology

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Abstract

In this paper we focus on the set-open topologies on the group $\mathcal{H}(X)$ of all self-homeomorphisms of a topological space X which yield continuity of both the group operations, product and inverse function. As a consequence, we make the more general case of Dijkstra's Theorem. In this case a homogeneously encircling family $\mathcal{B}$ consists of regular open sets and the closure of every set from $\mathcal{B}$ is contained in the finite union of connected sets from $\mathcal{B}$. Also we proved that the zero-cozero topology of $\mathcal{H}(X)$ is the relativisation to $\mathcal{H}(X)$ of the compact-open topology of $\mathcal{H}(\beta X)$ for any Tychonoff space X and every homogeneous zero-dimensional space X can be represented as the quotient space of a topological group with respect to a closed subgroup.

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1. Introduction

Let X be a topological space, $\mathcal{H}(X)$ the group of all self-homeomorphisms of X and $\mathbf{e} : (f, x) \in \mathcal{H}(X) \times X \rightarrow f(x) \in X$ the evaluation function. Since $\mathcal{H}(X)$ is a group with respect to the usual composition of functions, we consider those set-open topologies on $\mathcal{H}(X)$ which yield continuity of both the group operations, product and inverse function, and, at the same time, yield continuity of the evaluation function $\mathbf{e}$. In other words, we will focus on set-open topologies which make $\mathcal{H}(X)$ as a topological group and the evaluation function $\mathbf{e}$ as a group action of $\mathcal{H}(X)$ on X . Following [2], [3], a topology on $\mathcal{H}(X)$ which makes the evaluation function as a continuous function is called *admissible*. Admissible group topologies on $\mathcal{H}(X)$ are those ones which determine a group action of $\mathcal{H}(X)$ on X .

In [2] R. Arens proved that if a Hausdorff space X is noncompact, locally compact, and locally connected, then $\mathcal{H}(X)$ is a topological group because the compact-open topology coincides with the topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$, where αX is a Alexandroff one-point compactification of X .

In [10] J.J. Dijkstra proved that it is enough to required the property: any point of X has a connected compact neighbourhood.

Theorem 1.1. (*Dijkstra*) *Let X be a noncompact Hausdorff space. If every point in X has a neighborhood that is a continuum, then the compact-open topology on $\mathcal{H}(X)$ coincides with the group topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$.*

A. Di Concilio emphasis that local compactness of X is equivalent to the family of compact sets of X being a *boundedness* of X , [13], which, jointly any EF-proximity of X ,

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gives a local proximity space [14]. As a consequence, A.Di Concilio make this particular case to fall within the more general one in which compact sets are substituted with bounded sets in a local proximity space, while the property : *any point of X has a compact connected neighbourhood* is replaced by the following one: (*) for each non empty bounded set B there exists a finite number of connected bounded sets $B_1, \dots, B_n$ such that $B <<_\delta \text{int}(B_1) \cup \dots \cup \text{int}(B_n)$.

Theorem 1.2. (Theorem 5.1 in [7]) *If $(X, \mathcal{B}, \delta)$ is a local proximity space with the property (*) and any homeomorphism of X preserves both boundedness and proximity, then the topology of uniform convergence on bounded sets derived from the unique totally bounded uniformity associated with δ is an admissible group topology on $\mathcal{H}(X)$.*

Whenever $(X, \mathcal{B}, \delta)$ is a local proximity space, then the subcollection of $\mathcal{B}$ of all closed bounded subsets of X is a closed, hereditarily closed network of X .

We consider the family of $\mathcal{B}$ -open topologies which make $\mathcal{H}(X)$ as a topological group, but the family $\mathcal{B}$ is not necessarily a hereditarily closed network of X .

As a consequence, we make the more general case of Dijkstra's Theorem. In this case a homogeneously encircling family $\mathcal{B}$ consists of regular open sets and the closure of every set from $\mathcal{B}$ is contained in the finite union of connected sets from $\mathcal{B}$.

2. Set-Open Topology

The set-open topology is a generalization of the compact-open topology and the topology of pointwise convergence. This topology was first introduced by Arens and Dugundji [3]. Now the set-open topology effective is used in investigation of the topological properties of function spaces and the groups of all self-homeomorphisms of topological spaces [1, 5, 8, 9, 16, 17, 18, 19, 20, 22].

Definition 2.1. Let $\mathcal{B}$ be a family of non-empty subsets of a topological space X . The $\mathcal{B}$ -open topology on $\mathcal{H}(X)$ is a topology generated by the sets of the form $[U, V] = \{h \in \mathcal{H}(X) : h(U) \subset V\}$ where $U \in \mathcal{B}$ and V is open. The $\overline{\mathcal{B}}$ -open topology is a topology generated by the sets of the form $[\overline{U}, V]$.

We will denote as $\mathcal{H}_{\mathcal{B}}(X)$ and $\mathcal{H}_{\overline{\mathcal{B}}}(X)$ the set of all homeomorphisms of X equipped with the $\mathcal{B}$ -open topology and $\overline{\mathcal{B}}$ -open topology respectively. In particular, if $\mathcal{B}$ is a family of all compact subsets of X , then $\mathcal{H}_c(X)$ is the set of all homeomorphisms of X equipped with the compact-open topology.

Let $\mathcal{B}'$ be a minimal family containing $\mathcal{B}$ and closed under finite unions. Then $\mathcal{H}_{\mathcal{B}'}(X)$ is homeomorphic to $\mathcal{H}_{\mathcal{B}}(X)$ and $\mathcal{H}_{\overline{\mathcal{B}'}}(X)$ is homeomorphic to $\mathcal{H}_{\overline{\mathcal{B}}}(X)$ so we can always consider the family $\mathcal{B}$ be closed under finite unions. It is straightforward that the sets $[U, h(U)]$ with $U \in \mathcal{B}$ and $h \in \mathcal{H}(X)$ constitute the subbase of $\mathcal{H}_{\mathcal{B}}(X)$.

Note that the evaluation function $e : (f, x) \in \mathcal{H}(X) \times X \rightarrow f(x) \in X$ is continuous in a $\mathcal{B}$ -open topology when $\mathcal{B}$ is a base and it is continuous in a $\overline{\mathcal{B}}$ -open topology when $\mathcal{B}$ is a base and X is a regular space.

Recall that a family $\mathcal{B}$ of non-empty closed subsets of X containing all singletons and closed under finite unions is called a *Urysohn family* if whenever $A \in \mathcal{B}$ and a non-empty closed subset C are disjoint there exists $B \in \mathcal{B}$ such that $A \subset \text{int} B \subset B \subset X \setminus C$. For any family $\mathcal{B}$ of subsets of X , we will denote as $\mathcal{B}_H := \{h(A) : A \in \mathcal{B} \text{ and } h \in \mathcal{H}(X)\}$ the minimal family invariant under homeomorphisms containing $\mathcal{B}$ (see [8]).

In ([8], Lemma 2.1) A. Di Concilio proved the following result.

Proposition 2.1. Let $\mathcal{B}$ is a Urysohn family, then $\mathcal{B}_H$ is in turn a Urysohn family and $\mathcal{H}_{\mathcal{B}_H}(X)$ is a semigroup.

Definition 2.2. We will call $\mathcal{B}$ a *homogeneously encircling family* in X when $\mathcal{B}$ is a family of open in X sets with following properties.

- (a) $h(V) \in \mathcal{B}$ for every $V \in \mathcal{B}$ and every homeomorphism $h \in \mathcal{H}(X)$;
- (b) for every $V, W \in \mathcal{B}$ inclusion $\overline{V} \subset W$ implies $W \setminus \overline{V} \in \mathcal{B}$;
- (c) for every $B \in \mathcal{B}$ there is $B' \in \mathcal{B}$ such that $\overline{B} \subset B'$.

Obviously that if a family $\mathcal{B}$ satisfying condition (a) then $\mathcal{H}_{\mathcal{B}}(X)$ is a *paratopological group*, i.e. composition of homeomorphisms is a continuous operation.

Space $\mathcal{H}_{\overline{\mathcal{B}}}(X)$ is a paratopological group when $\mathcal{B}$ is a homogeneously encircling family satisfying the next condition:

- (d) for every $B, B' \in \mathcal{B}$ such that $\overline{B} \subset B'$ there is $B'' \in \mathcal{B}$ such that $\overline{B} \subset B'' \subset \overline{B''} \subset B'$.

Definition 2.3. We will call $\mathcal{B}$ *Urysohn homogeneously encircling family* in X when $\mathcal{B}$ is a family of open sets satisfying conditions (a), (b), (c) and (d).

It is well known that if X is a compact Hausdorff space, then $\mathcal{H}(X)$ with the compact-open topology is a topological group. In general, the compact-open topology does not provide continuity of the inverse function. For non-compact case, we have the Dijkstra Theorem 1.1 for the family $\mathcal{B}$ of all compact subsets of X and the Theorem 1.2 of Di Consilio (a generalization of Dijkstra's theorem) for the family $\mathcal{B}$ of all closed bounded subsets of X where $\mathcal{B}$ is a closed, hereditarily closed network of X .

Note that there exists a homogeneously encircling family $\mathcal{B}$ in X such that the $\mathcal{B}$ -open topology on $\mathcal{H}(X)$ is finer than the compact-open topology.

Let $X = \{0\} \cup \{\pm 1/n : n \in \mathbb{N}\}$ with usual topology of real line and let $\mathcal{B}$ be the family of all open subsets of X . We state that $\mathcal{H}_{\mathcal{B}}(X) > \mathcal{H}_c(X)$.

Let $V = \{1/n : n \in \mathbb{N}\}$ and id is a identical map. Suppose that $\text{id} \in \bigcap_{i=1}^p [K_i, U_i] \subset [V, V]$ for some compact sets K_i and open sets U_i . Let number j be so that $0 \in K_i$ for $i < j$ and $0 \notin K_i$ for $i \geq j$. There exists number $r \in \mathbb{N}$ such that for every $n > r$ $\pm 1/n \in U_i$ for $i < j$ and $\pm 1/n \notin K_i$ for $i \geq j$. Let h be a mapping defined as follows: $h(x) = x$ for any $x \geq 1/r$ and $h(x) = -x$ otherwise. It is easy to see that h is a homeomorphism and $h \in \bigcap_{i=1}^p [K_i, U_i]$ but $h \notin [V, V]$.

A subset W of X is called a regular open set provided $W = \text{Int}_X(\text{cl}_X(W))$.

Theorem 2.1. Let homogeneously encircling family $\mathcal{B}$ consists of regular open sets and the closure of every set from $\mathcal{B}$ is contained in the finite union of connected sets from $\mathcal{B}$. Then $\mathcal{H}_{\mathcal{B}}(X)$ is a topological group.

Proof. It suffices to prove that inversion $h \mapsto h^{-1}$ is a continuous map. Let $h^{-1} \in [U, h^{-1}(U)]$ where $U \in \mathcal{B}$. We will find an open neighborhood V of h such that $V^{-1} \subset [U, h^{-1}(U)]$, i.e. $g^{-1}(x) \in h^{-1}(U)$ for $g \in V$ and $x \in U$, hence, $g(x) \notin U$ for $g \in V$ and $x \notin h^{-1}(U)$. First, choose an open set $C = \bigcup_a C_a \supset h^{-1}(\overline{U})$ from family $\mathcal{B}$ where $\{C_a\}$ is a finite set of disjoint components. We can find in $\mathcal{B}$ sets C' and C'' such that $\overline{C} \subset C' \subset \overline{C'} \subset C''$. Let $V = [h^{-1}(U), U] \cap [C' \setminus h^{-1}(\overline{U}), h(C' \setminus h^{-1}(\overline{U}))] \cap \bigcap_a [C_a, h(C_a)] \cap [C'' \setminus \overline{C}, h(C'' \setminus \overline{C})]$ and $g \in V$. If $g(x) \in h(C_a \cap h^{-1}(U))$ for some $x \notin C''$ then $g(x)$ is contained in connected set $h(C_a)$ with $g(C_a)$ because $g \in V \subset [C_a, h(C_a)]$. Hence $g^{-1}(h(C_a))$ is a connected set that contains x and C_a which implies that $g^{-1}(h(C_a)) \cap (\overline{C'} \setminus C') \neq \emptyset$. But this fact means that $g(\overline{C'} \setminus C') \cap h(C_a) \neq \emptyset$. It is contradiction due to the fact that $g \in V \subset [C'' \setminus \overline{C}, h(C'' \setminus \overline{C})]$ and $h(C_a) \subset h(C)$. Obviously $g(x) \notin U$ for every $x \in C'' \setminus h^{-1}(\overline{U})$.

We conclude that $g(\overline{U}) \subseteq h(\overline{U})$ or equally $\overline{g(U)} \subseteq \overline{h(U)}$. The latter implies $g(U) \subseteq h(U)$ because U is a regular open set. $\square$

Theorem 2.2. *Let Urysohn homogeneously encircling family $\mathcal{B}$ be such that the closure of every set from $\mathcal{B}$ is contained in the finite union of connected sets from $\mathcal{B}$. Then $\mathcal{H}_{\overline{\mathcal{B}}}(X)$ is a topological group.*

Proof. The proof is the same as in the Theorem 2.1. $\square$

The following example shows that Theorem 2.1 is a generalization of the Dijkstra and Di Concilio Theorems.

Let $Z = \mathbb{N} \times [0, 1] = \bigcup \{\mathbb{I}_n : n \in \mathbb{N}\}$, $X = \alpha Z = Z \cup \{\alpha\}$ where αZ is a Alexandroff one-point compactification of Z and $\mathcal{B}$ is the family of all finite unions of intervals $\mathbb{I}_n = [0, 1]_n$.

Since $h(\alpha) = \alpha$ for any $h \in \mathcal{H}(X)$, then $\mathcal{B}$ is a homogeneously encircling family consists of regular open sets and the closure of every set from $\mathcal{B}$ is contained in the finite union of connected sets from $\mathcal{B}$. By Theorem 2.1, $\mathcal{H}_{\mathcal{B}}(X)$ is a topological group. Note that the point α does not have a neighborhood continuum and, moreover, $\mathcal{B}$ is not a hereditarily closed network of X .

A topological space X is called semiregular provided that for each $U \in X$ and $x \in U$ there exists a regular open set V in X such that $x \in V \subseteq U$.

Proposition 2.2. Let X be a locally compact (semiregular) space and $\mathcal{B}$ consists of all (regular open) open sets with compact closure. Then $\mathcal{H}_{\mathcal{B}}(X) \geq \mathcal{H}_c(X)$ and $\mathcal{H}_{\overline{\mathcal{B}}}(X)$ is homeomorphic to $\mathcal{H}_c(X)$.

Proof. Clearly that $\mathcal{B}$ is a Urysohn homogeneously encircling family. Let $h \in [K, U]$ where K is a compact and U is an open set. There is a regular open set V such that $K \subset V \subset \overline{V} \subset h^{-1}(U)$ and $\overline{V}$ is compact. It immediately follows that $h \in [V, U] \subset [K, U]$ and $h \in [\overline{V}, U] \subset [K, U]$. $\square$

Proposition 2.3. $\mathcal{H}_{\mathcal{B}}(X) \geq \mathcal{H}_{\overline{\mathcal{B}}}(X)$ for every Urysohn homogeneously encircling hereditarily open family $\mathcal{B}$.

Proof. Let $h \in [\overline{V}, U]$ where $V \in \mathcal{B}$. There exists set $W \in \mathcal{B}$ such that $W \supset \overline{V}$. Obviously that $h \in [W \cap h^{-1}(U), U] \subset [\overline{V}, U]$. $\square$

Corollary 2.1. (Dijkstra's Theorem 1.1) Note that if every point from X has a neighborhood which is continuum. Then $\mathcal{H}_c(X)$ is a topological group.

Proof. Clearly that X is a locally compact space. Let $\mathcal{B}$ be a family of all open sets with compact closure. It follows from Proposition 2.2 that $\mathcal{H}_{\overline{\mathcal{B}}}(X)$ is homeomorphic to $\mathcal{H}_c(X)$. The fact that every compact set is contained in a finite union of continua and, by Theorem 2.2, we have a complete proof of the corollary. $\square$

Let $X = \mathbb{R}^2$ and $\mathcal{B}$ be a family of all bounded open sets. For $x \in X$ let the family of the sets $A_{x, \alpha, \beta} = \{y \in X : \alpha < \arg(y - x) < \beta\}$ where $\alpha, \beta \in [0, 2\pi]$ be denoted as $\mathcal{A}_x$. It is straightforward that the family $\mathcal{B}_{x_1, \dots, x_n} = \mathcal{B} \cup \mathcal{A}_{x_1} \cup \dots \cup \mathcal{A}_{x_n}$ is a Urysohn homogeneously encircling family and the closure of every set from $\mathcal{B}_{x_1, \dots, x_n}$ is contained in the finite union of connected sets from $\mathcal{B}_{x_1, \dots, x_n}$. The latter implies that $\mathcal{H}_{\overline{\mathcal{B}_{x_1, \dots, x_n}}}(X)$ is a topological group due to the Theorem 2.2. It is easy to see that $\mathcal{H}_{\overline{\mathcal{B}_{x_1, \dots, x_n}}}(X) \leq \mathcal{H}_{\overline{\mathcal{B}_{y_1, \dots, y_m}}}(X)$ if and only if $\{x_1, \dots, x_n\} \subset \{y_1, \dots, y_m\}$. So we have an infinite lattice of group topologies on $\mathcal{H}(X)$.

Let $\mathcal{RO}$ be a family of all regular open subsets of X . It is known that $\mathcal{H}_{\mathcal{RO}}(X)$ is a topological group (Theorem 4 in [21]). Then for every family $\mathcal{B}$ satisfying conditions from Theorem 2.1, $\mathcal{H}_{\mathcal{B}}(X)$ is a topological group and $\mathcal{H}_{\mathcal{B}}(X) \leq \mathcal{H}_{\mathcal{RO}}(X)$.

3. Metrization Condition

A space X is a q -space if for every point $x \in X$ there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open neighbourhoods of x in X such that for every choice $x_n \in U_n$, the sequence $\{x_n : n \in \mathbb{N}\}$ has a cluster point.

D.Gauld and J. van Mill proved that a manifold M is metrizable if and only if $\mathcal{H}_c(M)$ is a q -space.

Theorem 3.1. (Theorem 4.2 in [12]) *For a manifold M , the following statements are equivalent:*

1. M is metrizable;
2. M is separable and metrizable;
3. $\mathcal{H}_c(M)$ is first countable;
4. $\mathcal{H}_c(M)$ is a q -space;
5. $\mathcal{H}_c(M)$ is metrizable;
6. $\mathcal{H}_c(M)$ is separable and metrizable.

We say that X is *strongly locally homogeneous* (abbreviated SLH) if it has a base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is an element $f \in \mathcal{H}(X)$ that is supported on B (that is, f is the identity outside B) and moves x to y . Clearly every manifold is SLH.

We are looking for conditions on $\mathcal{H}_{\overline{B}}(X)$ and the family $\mathcal{B}$ for a SLH space X that ensure that X is metrizable.

Recall that *strong development* for X is a sequence of open covers $\{V_n\}$ such that for every $x \in X$ and every neighborhood U of x there are open W and number i such that $x \in W \subset \text{St}(W, V_i) \subset U$.

Theorem 3.2. *Let X be SLH space and let $\mathcal{B}$ be Urysohn homogeneously encircling base. If $\mathcal{H}_{\overline{B}}(X)$ is metrizable topological group then X is metrizable.*

Proof. Let $\{U_n\}$ be a sequence of open symmetric neighborhoods of the identity map $e \in \mathcal{H}_{\overline{B}}(X)$ such that $U_{n+1}^3 \subset U_n$ for every n and $\{U_n\}$ is a local base at e . Let $V_n = \{\text{Int}(\gamma_x(U_n)) : x \in X\}$ where $\gamma_x : h \mapsto h(x)$. We claim that $\{V_n\}$ is a strong development for X which suffices by the Moore metrization Theorem.

Claim that $x \in \text{Int}(\gamma_x(U_n))$ for every $x \in X$ and every number n .

Let $U_n \supset \bigcap_{i=1}^k [\overline{P}_i, Q_i]$ and $x \in Q_i$ for $i < j$ and $x \notin Q_i$ for $i \geq j$. Since X is SLH, x has an open neighborhood W such that $W \subset Q_i$ for $i < j$ and $W \cap \overline{P}_i = \emptyset$ for $i \geq j$ and for any $y \in W$ there is homeomorphism h supported by W such that $h(x) = y$. So $x \in W \subset \text{int}(\gamma_x(U_n))$.

Let $U = \gamma_x(U_n)$, $W = \gamma_y(U_n)$ and $U \cap W \neq \emptyset$. For any $w \in W$ there are $z \in U \cap W$ and $f, g, h \in U_n$ such that $f(x) = z$, $g(y) = z$, $h(y) = w$. The observation $h \circ g^{-1} \circ f \in U_n^3 \subset U_{n-1}$ implies $\text{St}(\gamma_x(U_n), V_n) \subset \gamma_x(U_{n-1})$ i.e. V_n is a star refinement of V_{n-1} . It suffices to prove that $\bigcup_n V_n$ is a base of X . Let V be an open set and $x \in V$. There is $B \in \mathcal{B}$ such that $x \in B \subset \overline{B} \subset V$. So we can choose $U_n \subset [\overline{B}, V]$. It is immediate that $x \in \text{int}(\gamma_x(U_n)) \subset V$. $\square$

4. Zero-Cozero Topology

Let us recall that a set $F \subseteq X$ is a *zero set* if there exists a continuous function $f \in C(X, \mathbb{R})$ such that $F = \{x \in X : f(x) = 0\}$. A complement of a zero set is a *cozero set*.

Consider the family $\mathcal{P}$ of the sets of the form $[A, U] = \{h \in \mathcal{H}(X) : h(A) \subset U\}$ where A is a zero-set of X and U is a cozero set of X . Let the set $\mathcal{H}(X)$, equipped with *the zero-cozero topology* generated by $\mathcal{P}$, be denoted as $\mathcal{H}_{zc}(X)$. When A runs over all closed sets in X and U is open in X then we get *the closed-open topology*.

Since any self-homeomorphism of a Tychonoff space X continuously extends to the Stone-Ćech compactification βX of X , then $\mathcal{H}(X)$ embeds as a subgroup in $\mathcal{H}(\beta X)$. Thereby, the relativisation to $\mathcal{H}(X)$ of the compact-open topology of $\mathcal{H}(\beta X)$ is an admissible group topology.

Proposition 4.1. For a Tychonoff space X , $\mathcal{H}_{zc}(X)$ is the relativisation to $\mathcal{H}(X)$ of the compact-open topology of $\mathcal{H}(\beta X)$.

The proof is similar to the proof of Theorem 1.8 in [6] in the case of normal spaces. In fact, if X is normal space then the closed-open topology on $\mathcal{H}(X)$ coincides with the zero-cozero topology $\mathcal{H}(X)$. Then we get the next result of A. Di Concilio.

Corollary 4.1. (Theorem 1.8 in [6]) When X is T_4 , the relativization $\tau_{\beta X}$ to $\mathcal{H}(X)$ of the compact-open topology on $\mathcal{H}(\beta X)$ is the closed-open topology.

Corollary 4.2. If X is a Tychonoff space, then $\mathcal{H}_{zc}(X)$ with the zero-cozero topology is a topological group.

Important natural examples of homogeneous spaces are provided by quotients of topological groups with respect to closed subgroups. Though these quotients need not be topological groups themselves, they are always homogeneous topological spaces.

A natural question for consideration is the following one [4]: which homogeneous spaces can be represented as quotients of topological group with respect to closed subgroup? A partial answer to this question is given below.

Theorem 4.1. (N. Bourbaki). Every homogeneous zero-dimensional compact space X can be represented as the quotient space of a topological group with respect to a closed subgroup.

Proposition 4.2. (L.R. Ford in [11]) If a zero-dimensional T_1 -space X is homogeneous, then it is SLH.

Suppose that X is a space. A topology τ on the group $\mathcal{H}(X)$ will be called *acceptable* if it turns $\mathcal{H}(X)$ into a topological group such that the evaluation function is continuous with respect to the first variable, and for every open neighbourhood U of the neutral element e of $\mathcal{H}(X)$ and for each $b \in X$, there exists an open neighbourhood V of b such that $Id_V \subset U$, where $Id_V := \{h \in \mathcal{H}(X) : h(x) = x \text{ for each } x \in X \setminus V\}$.

The next statement is obvious, in view of Corollary 4.2 and of the definition of acceptable topologies.

Proposition 4.3. For a Tychonoff space X , the zero-cozero topology on the group $\mathcal{H}(X)$ is acceptable.

Proposition 4.4. (Proposition 3.5.14 in [4]) Suppose that X is a homogeneous SLH space, and $\mathcal{H}(X)$ is endowed with an acceptable topology. Then X is canonically homeomorphic to the quotient space $\mathcal{H}(X)/G_a$ where a is a point of X and G_a is the stabilizer of a in G , that is, $G_a = \{g \in G : g(a) = a\}$.

Proposition 4.5. Every homogeneous zero-dimensional space X can be represented as the quotient space of a topological group with respect to a closed subgroup.

Proof. By Proposition 4.2, the space X is SLH. By Proposition 4.3, the zero-cozero topology on the group $\mathcal{H}(X)$ is acceptable. Now it follows from Proposition 4.4 that X is homeomorphic to a quotient space of the group $\mathcal{H}_{zc}(X)$ with respect to a closed subgroup. $\square$

Remark 4.1. By Proposition 4.1, the zero-cozero topology on $\mathcal{H}_{zc}(X)$ can be represented as an induced topology of uniform convergence on the family of all compact subsets of βX on the space $\mathcal{H}_c(\beta X)$. Consequently, Proposition 4.5 is a special case of Theorem 4.1 of L.R. Ford [11] (see also [15]).

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