Network-Type and Measure-Functor Properties of the Space of Solutions of Nonlinear Volterra–Fredholm Integro-Differential Equations

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Abstract

We study topological and network-type properties of the solution space of a nonlinear Volterra–Fredholm integro-differential equation on a compact interval. The equation contains both a Volterra memory term and a Fredholm nonlocal term and is considered under continuity assumptions on all nonlinearities and kernels. The space of classical solutions is endowed with the topology inherited from the Banach space of continuous vector-valued functions with the uniform norm.

The first main result shows that the solution space is closed in the ambient Banach space. Consequently, it is a Polish space. A Polish space is a separable, completely metrizable topological space. We then prove a stronger structural fact: the solution space is $\sigma-$compact. Indeed, its intersection with every uniform norm ball is compact; this follows from uniform bounds for the integral terms, a uniform derivative estimate, and the Arzelà–Ascoli theorem. These compactness properties remain valid in the genuinely nonlocal Volterra–Fredholm setting despite the presence of memory and global feedback.

As consequences of the Polish and second countable structure, the solution space has a countable network, a countable $cs-$network, a countable cs^*-network, a countable $cn-$network, and a countable $ck-$network. We further investigate measure-functor constructions. The space of all Borel probability measures on the solution space, equipped with the weak topology, is Polish and hence possesses all of the above countable network-type properties. The same is true for its finite-support subspaces. For each compact bounded stratum of the solution space, the spaces of probability measures and idempotent probability measures are compact metrizable spaces; therefore their network, $cs-$network, cs^*-network, $cn-$network, and $ck-$network are countable. Functorial consequences under continuous transformations of solution spaces are also recorded.

The results extend the topological program previously developed for spaces of solutions of ordinary differential equations to a genuinely nonlocal Volterra–Fredholm setting and reveal additional Polish and compactness features produced by the integral structure.

Keywords: integro-differential equation; Volterra operator; Fredholm operator; solution space; Polish space; $\sigma-$compact space; network; $cs-$network; cs^*-network; $cn-$network; $ck-$network; probability measure; idempotent probability measure.
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1. Introduction. Problem statement

The study of spaces of solutions of differential equations from the viewpoint of general topology has developed into a useful meeting point of differential equations, functional analysis, and functorial topology [1]–[13] of considering only existence, uniqueness, or asymptotic properties of an individual solution, one regards the whole family of solutions as a topological subspace of a suitable function space and investigates its cardinal, convergence, compactness, homotopy, and measure-theoretic properties.

For ordinary differential equations with continuous right-hand sides, this approach leads naturally to solution spaces contained in $C(I, \mathbb{R}^m)$, where I is a compact interval [10], [13]. The work [13] has considered compactness, density, weight, network weight, tightness, connectedness, contractibility, density-type invariants, and several network-type properties of such solution spaces. In particular, the existence of countable networks, $cs-$networks, cs^*-networks, $cn-$networks, and $ck-$networks follows from the metrizable and separable nature of the ambient function space.

The purpose of the present paper is to pass from ordinary differential equations to a genuinely nonlocal class of integro-differential equations. We consider the nonlinear Volterra–Fredholm equation

$$x'(t) = F(t, x(t), (\mathcal{V}x)(t), (\mathcal{W}x)(t)), \qquad t \in I = [a, b], \tag{1.1}$$

where

$$(\mathcal{V}x)(t) = \int_a^t K_V(t, s, x(s))\, ds \tag{1.2}$$

is a Volterra memory term and

$$(\mathcal{W}x)(t) = \int_a^b K_F(t, s, x(s))\, ds \tag{1.3}$$

is a Fredholm nonlocal term. Here

$$F : I \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

$$K_V : D_V \times \mathbb{R}^m \to \mathbb{R}^m, \qquad D_V = \big\{(t, s) \in I^2 : a \le s \le t \le b\big\},$$

and

$$K_F : I^2 \times \mathbb{R}^m \to \mathbb{R}^m$$

are continuous.

This class contains ordinary differential equations as a degenerate case and also includes nonlinear equations with memory and global integral feedback. The integral terms make the equation analytically different from an ODE, but they also produce useful compactness mechanisms. In particular, on a uniformly bounded family of candidate solutions, continuity of the kernels gives uniform bounds for both integral operators, while continuity of F then yields a uniform derivative bound. This allows the Arzelà–Ascoli theorem to be used directly on bounded solution strata.

The first objective of this paper is therefore not merely to repeat the metric argument known for ODE solution spaces, but to identify a stronger topological structure of the solution

set. We prove that the space $\mathcal{S}$ of classical solutions of (1.1) is a closed subspace of $C(I, \mathbb{R}^m)$ and hence is Polish. We then prove that

$$\mathcal{S} = \bigcup_{r=1}^{\infty} \mathcal{S}_r, \qquad \mathcal{S}_r = \{x \in \mathcal{S} : \|x\|_\infty \leq r\},$$

with every $\mathcal{S}_r$ compact. Thus $\mathcal{S}$ is $\sigma-$compact.

These facts immediately strengthen the network-type conclusions. A Polish space is a separable, completely metrizable topological space. Since a Polish space is second countable, a single countable base is simultaneously a countable network, $cs-$network, cs^*-network, $cn-$network, and $ck-$network. This avoids the need to construct a separate family for each network notion.

The second objective concerns spaces of measures over the solution space. For a Polish space X, the space $\mathcal{P}(X)$ of Borel probability measures with the weak topology is itself Polish. We apply this fact to $X = \mathcal{S}$, obtaining all countable network-type properties for the entire probability-measure space, not only for finite-support measures. For the compact solution strata $\mathcal{S}_r$, we also consider the functor $\mathcal{I}$ of idempotent probability measures. Since each $\mathcal{S}_r$ is compact metrizable, $\mathcal{I}(\mathcal{S}_r)$ is compact metrizable and therefore possesses the same countable network-type properties.

The resulting theory thus has two layers. The first is analytic: continuity of the Volterra–Fredholm operators yields closedness and compactness of bounded solution families. The second is topological: Polishness and metrizability automatically transfer strong countability properties to the solution space and to natural measure functors over it.

The paper is organized as follows. Section 2 recalls the network notions and the measure constructions needed below. Section 3 introduces the Volterra–Fredholm equation and proves continuity of the integral operators. Section 4 establishes the closed Polish and $\sigma-$compact structure of the solution space. Section 5 derives the network-type consequences. Section 6 studies probability measures. Section 7 treats idempotent measures on compact solution strata. Section 8 records functorial consequences, and Section 9 contains a discussion of the scope and possible extensions of the results.

2. Preliminaries

Throughout the paper, $I = [a, b] \subset \mathbb{R}$ is a fixed compact interval and $C(I, \mathbb{R}^m)$ is equipped with the uniform norm

$$\|x\|_\infty = \sup_{t \in I} \|x(t)\|.$$

This is a separable Banach space and therefore a Polish space.

Definition 2.1. A family $\mathcal{N}$ of subsets of a topological space X is called a network if for every $x \in X$ and every neighborhood U of x there exists $N \in \mathcal{N}$ such that

$$x \in N \subset U.$$

Definition 2.2. A family $\mathcal{N}$ is a $cs-$network at $x \in X$ if for every sequence (x_j) converging to x and every neighborhood U of x there exists $N \in \mathcal{N}$ such that $N \subset U$ and

$$\{x\} \cup \{x_j : j \geq j_0\} \subset N$$

for some j_0.

Definition 2.3. A family $\mathcal{N}$ is a cs^*-network at x if, whenever $x_j \to x$ and U is a neighborhood of x, there are a subsequence (x_{j_k}) and $N \in \mathcal{N}$ such that

$$\{x_{j_k} : k \in N\} \subset \mathcal{N} \subset U.$$

Definition 2.4. A family $\mathcal{N}$ is a *$cn-$network* at x if, for every neighborhood O_x of x, the set

$$\cup\{N \in \mathcal{N} : x \in N \subset O_x\}$$

is a neighborhood of x.

Definition 2.5. A family $\mathcal{N}$ is a $ck-$network at x if, for every neighborhood O_x of x, there exists a neighborhood $U_x \subset O_x$ such that for each compact $K \subset U_x$ there is a finite subfamily $\mathcal{F} \subset \mathbb{N}$ satisfying

$$x \in \cap\mathcal{F}, \qquad K \subset \cup\mathcal{F} \subset O_x.$$

Every countable base is simultaneously a countable network, a countable $cs-$network, a countable cs^*-network, a countable $cn-$network, and a countable $ck-$network.

For a separable metric space X, let $\mathcal{P}(X)$ denote the space of Borel probability measures on X endowed with the weak topology. If X is Polish, then $\mathcal{P}(X)$ is Polish. For $n \in \mathbb{N}$, we put

$$\mathcal{P}_n(X) = \big\{\mu \in \mathcal{P}(X) : |\operatorname{supp}\mu| \le n\big\}.$$

If X is a compact Hausdorff space, let $\mathcal{I}(X)$ be the space of idempotent probability measures (Maslov measures) on X. A functional $\mu : C(X) \to \mathbb{R}$ is idempotent if

$$\mu(c_X) = c, \qquad \mu(c \odot \varphi) = c \odot \mu(\varphi), \qquad \mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi),$$

where

$$u \oplus v = \max\big\{u, v\big\}, \qquad u \odot v = u + v.$$

The space $\mathcal{I}(X)$ is endowed with the topology of pointwise convergence. For compact metrizable X, the space $\mathcal{I}(X)$ is compact metrizable.

3. The Volterra–Fredholm equation

Let the functions F, K_V, and K_F satisfy the continuity assumptions stated in the introduction. We define the operators $\mathcal{V}$ and $\mathcal{W}$ by (1.2) and (1.3).

A function $x \in C^1(I, \mathbb{R}^m)$ is called a classical solution of (1.1) if the equation holds for every $t \in I$.

Let

$$\mathcal{S} = \big\{x \in C(I, \mathbb{R}^m) : x \in C^1(I, \mathbb{R}^m) \text{ and } x \text{ satisfies (1.1)}\big\}.$$

We equip $\mathcal{S}$ with the subspace topology inherited from $C(I, \mathbb{R}^m)$.

Lemma 3.1. The maps

$$\mathcal{V}, \mathcal{W} : C(I, \mathbb{R}^m) \to C(I, \mathbb{R}^m)$$

are continuous with respect to the uniform norm.

Proof. Let $x_j \to x$ uniformly as $j \to \infty$. Then the sequence (x_j) is uniformly bounded: there exists $R > 0$ such that

$$x_j(I) \cup x(I) \subset \overline{B}_R(0)$$

for all sufficiently large j.

The function K_V is uniformly continuous on the compact set

$$D_V \times \overline{B}_R(0).$$

Hence,

$$\sup_{(t,s)\in D_V} \|K_V(t,s,x_j(s)) - K_V(t,s,x(s))\| \longrightarrow 0 \text{ as } j \to \infty.$$

Therefore,

$$\|\mathcal{V}x_j - \mathcal{V}x\|_\infty \leq (b-a) \sup_{(t,s)\in D_V} \|K_V(t,s,x_j(s)) - K_V(t,s,x(s))\| \to 0 \text{ as } j \to \infty.$$

The proof for $\mathcal{W}$ is identical, using the compact set $I^2 \times \overline{B}_R(0)$. □

Lemma 3.2. Define

$$\mathcal{G}(x)(t) = F(t, x(t), (\mathcal{V}x)(t), (\mathcal{W}x)(t)).$$

Then

$$\mathcal{G} : C(I, \mathbb{R}^m) \to C(I, \mathbb{R}^m)$$

is continuous in the uniform norm.

Proof. Let $x_j \to x$ uniformly as $j \to \infty$. By Lemma 3.1,

$$\mathcal{V}x_j \to \mathcal{V}x, \qquad \mathcal{W}x_j \to \mathcal{W}x$$

uniformly. All values of the four arguments of F therefore remain in a common compact subset of $I \times (\mathbb{R}^m)^3$. Uniform continuity of F on this compact set yields

$$\|\mathcal{G}(x_j) - \mathcal{G}(x)\|_\infty \to 0 \text{ as } j \to \infty.$$

□

4. Polish and compactness properties of the solution space

The differential equation (1.1) is equivalent to the integral identity equation

$$x(t) = x(a) + \int_a^t F(\tau, x(\tau), (\mathcal{V}x)(\tau), (\mathcal{W}x)(\tau))\, d\tau.$$

Theorem 4.1. The solution space $\mathcal{S}$ is closed in $C(I, \mathbb{R}^m)$. Consequently, $\mathcal{S}$ is a Polish space.

Proof. Let $x_j \in \mathcal{S}$ and suppose

$$x_j \to x \text{ as } j \to \infty$$

uniformly in $C(I, \mathbb{R}^m)$. Since every x_j is a solution,

$$x_j(t) = x_j(a) + \int_a^t \mathcal{G}(x_j)(\tau)\, d\tau.$$

By Lemma 3.2,

$$\mathcal{G}(x_j) \to \mathcal{G}(x)$$

uniformly. Passing to the limit gives

$$x(t) = x(a) + \int_a^t \mathcal{G}(x)(\tau)\, d\tau.$$

The integrand is continuous, hence $x \in C^1(I, \mathbb{R}^m)$ and

$$x'(t) = \mathcal{G}(x)(t).$$

Thus $x \in \mathcal{S}$, proving that $\mathcal{S}$ is closed.

The space $C(I, \mathbb{R}^m)$ is Polish. A closed subspace of a Polish space is Polish, so $\mathcal{S}$ is Polish. □

For $r \in \mathcal{N}$ define

$$\mathcal{S}_r = \{x \in \mathcal{S} : \|x\|_\infty \le r\}.$$

Theorem 4.2. For every $r \in \mathcal{N}$, the set $\mathcal{S}_r$ is compact in $C(I, \mathbb{R}^m)$.

Proof. Fix r. Since K_V is continuous on the compact set

$$D_V \times \overline{B}_r(0),$$

there exists $M_{V,r} > 0$ such that

$$\|K_V(t, s, u)\| \le M_{V,r}.$$

Similarly, there exists $M_{F,r} > 0$ such that

$$\|K_F(t, s, u)\| \le M_{F,r}$$

on $I^2 \times \overline{B}_r(0)$.

Hence for every $x \in \mathcal{S}_r$,

$$\|(\mathcal{V}x)(t)\| \le (b-a)M_{V,r},$$

and

$$\|(\mathcal{W}x)(t)\| \le (b-a)M_{F,r}.$$

Therefore, all arguments of F belong to the compact set

$$I \times \overline{B}_r(0) \times \overline{B}_{(b-a)M_{V,r}}(0) \times \overline{B}_{(b-a)M_{F,r}}(0).$$

By continuity of F, there exists $M_r > 0$ such that

$$\|x'(t)\| \le M_r$$

for all $x \in \mathcal{S}_r$ and $t \in I$.

Thus $\mathcal{S}_r$ is uniformly bounded and equi-Lipschitz:

$$\|x(t) - x(s)\| \le M_r|t - s|.$$

By the Arzelà–Ascoli theorem, $\mathcal{S}_r$ is relatively compact in $C(I, \mathbb{R}^m)$. It is closed by Theorem 4.1 and by the closedness of the uniform norm ball. Hence $\mathcal{S}_r$ is compact. □

Corollary 4.1. The solution space $\mathcal{S}$ is $\sigma-$compact:

$$\mathcal{S} = \bigcup_{r=1}^{\infty} \mathcal{S}_r.$$

Remark 4.1. No uniqueness assumption for (1.1) is needed in Theorems 4.1–4.2. The results concern the full family of all classical solutions on the fixed compact interval.

5. Network-type properties

Theorem 5.1. The space $\mathcal{S}$ has

1. a countable network;
2. a countable $cs-$network;
3. a countable cs^*-network;
4. a countable $cn-$network;
5. a countable $ck-$network.

Proof. By Theorem 4.1, $\mathcal{S}$ is Polish and hence second countable. Let $\mathcal{B}$ be a countable base for $\mathcal{S}$.

Every base is a network. If $x_j \to x$ as $j \to \infty$ and U is a neighborhood of x, choose $B \in \mathcal{B}$ with

$$x \in B \subset U.$$

Eventually $x_j \in B$, so $\mathcal{B}$ is a $cs-$network. The same argument immediately gives the cs^*-property.

For the $cn-$property, the union of all $B \in \mathcal{B}$ satisfying $x \in B \subset U$ is an open neighborhood of x contained in U.

Finally, every base is a $ck-$network. If $\mathcal{B}$ is a base and O_x is a neighborhood of x, we choose $x \in B \subset O_x$, $B \in \mathcal{B}$, we set $U_x = B$, and for any compact $K \subset U_x$ we take $\mathcal{F} = \{B\}$. Then automatically, we have $x \in B$, $K \subset B \subset O_x$. □

Corollary 5.1. For every $r \in \mathcal{N}$, the compact solution stratum $\mathcal{S}_r$ has all five countable network-type properties listed in Theorem 5.1.

6. Probability measures on the solution space

Let $\mathcal{P}(\mathcal{S})$ denote the space of all Borel probability measures on $\mathcal{S}$ with the weak topology.

Theorem 6.1. *The space $\mathcal{P}(\mathcal{S})$ is Polish. Consequently, it has a countable network, a countable $cs-$network, a countable cs^*-network, a countable $cn-$network, and a countable $ck-$network.*

Proof. By Theorem 4.1, $\mathcal{S}$ is Polish. A standard theorem on probability measures on Polish spaces states that $\mathcal{P}(\mathcal{S})$, endowed with the weak topology, is again Polish. Hence it is second countable. The conclusion follows exactly as in Theorem 5.1. □

Corollary 6.1. For every $n \in \mathcal{N}$, the finite-support subspace

$$\mathcal{P}_n(\mathcal{S}) = \{\mu \in \mathcal{P}(\mathcal{S}) : |\operatorname{supp}\mu| \leq n\}$$

has all five countable network-type properties.

Proof. Every subspace of a second countable space is second countable. □

Corollary 6.2. For every $r \in \mathcal{N}$, the space $\mathcal{P}(\mathcal{S}_r)$ is compact metrizable and has all five countable network-type properties.

7. Idempotent probability measures on compact solution strata

The full idempotent probability-measure functor is naturally formulated on compact Hausdorff spaces. The compact sets $\mathcal{S}_r \neq \emptyset$ obtained in Theorem 4.2 therefore provide canonical domains for this construction.

Theorem 7.1. For every $r \in \mathcal{N}$ such that $\mathcal{S}_r \neq \emptyset$, the space $\mathcal{I}(\mathcal{S}_r)$ is compact metrizable.

Proof. By Theorem 4.2, $\mathcal{S}_r$ is compact metrizable. The metrization theorem for idempotent probability measures on compact metric spaces implies that $\mathcal{I}(\mathcal{S}_r)$ is metrizable in its pointwise convergence topology. The idempotent probability-measure functor maps compacta to compacta, so $\mathcal{I}(\mathcal{S}_r)$ is compact. □

Corollary 7.1. For every $r, n \in \mathcal{N}$, both $\mathcal{I}(\mathcal{S}_r)$ and its finite-support subspace

$$\mathcal{I}_n(\mathcal{S}_r) = \{\mu \in \mathcal{I}(\mathcal{S}_r) : |\operatorname{supp}\mu| \leq n\}$$

have a countable network, a countable $cs-$network, a countable cs^*-network, a countable $cn-$network, and a countable $ck-$network.

8. Functorial consequences

Let $\mathcal{S}$ and $\mathcal{T}$ be solution spaces of two Volterra–Fredholm integro-differential equations of the form (1.1). Suppose the map

$$h : \mathcal{S} \to \mathcal{T}$$

is continuous. Then the push-forward map

$$\mathcal{P}(h) : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{T}), \qquad \mathcal{P}(h)(\mu)(A) = \mu(h^{-1}(A)),$$

is continuous in the weak topology.

If, in addition, h maps a compact solution stratum $\mathcal{S}_r$ into a compact solution stratum $\mathcal{T}_q$, then the induced map

$$\mathcal{I}(h) : \mathcal{I}(\mathcal{S}_r) \to \mathcal{I}(\mathcal{T}_q), \qquad \mathcal{I}(h)(\mu)(\varphi) = \mu(\varphi \circ h)$$

is continuous.

Proposition 8.1. The probability-measure construction preserves the Polish character of the solution spaces considered above, while both the probability-measure and idempotent probability-measure constructions preserve compact metrizability on compact solution strata.

Proof. The first assertion follows from the general theorem that the probability space over a Polish space is Polish. The second follows from the compact-metrizable character of $\mathcal{S}_r$ and the standard functorial properties of $\mathcal{P}$ and $\mathcal{I}$ on compact metrizable spaces. □

9. Discussion

The passage from ordinary differential equations to Volterra–Fredholm integro-differential equations changes the analytic content of the solution-space problem in an essential way. For an ODE, the basic network conclusions may be read directly from the fact that the solution set is a subspace of a separable metric function space. The same observation remains true for (1.1), but by itself it would produce only a formal extension.

The stronger point is that the integral equation permits a direct proof that the solution space is closed. Uniform convergence of solutions implies uniform convergence of the Volterra and Fredholm terms, because the kernels are uniformly continuous on compact subsets. Passing to the limit in the integral form of the equation then shows that the uniform limit is again a classical solution. This yields completeness of the solution space in a compatible metric and hence Polishness.

A second structural gain is the compactness of the bounded strata $\mathcal{S}_r$. The boundedness of the solutions forces the kernel values to remain in compact sets, which bounds both integral terms. The right-hand side of the differential equation is then uniformly bounded, yielding a uniform Lipschitz constant for the whole stratum. Arzelà–Ascoli therefore converts a simple a priori bound into compactness. As a consequence, the solution space is not merely separable and metrizable but $\sigma-$compact Polish.

This distinction is important for the measure constructions. For the full solution space, Polishness gives a well-behaved space of all Borel probability measures, a result stronger than statements restricted to finite-support measures. On the other hand, idempotent probability measures are most naturally treated over compacta. The compact solution strata provide exactly the appropriate objects for the idempotent-measure functor, without imposing an artificial global compactness assumption on the whole solution space.

The network-type conclusions themselves then become transparent. Once a space is second countable, a countable base simultaneously provides all five network structures considered in this paper. This also avoids technical difficulties that can arise when one tries to prove the $ck-$network property from a family of metric balls centered at a dense set. The compact-cover condition in the definition of a $ck-$network is most cleanly handled by working with a base.

The continuity assumptions imposed here are deliberately minimal for the topological statements. No global Lipschitz condition and no uniqueness theorem are required. If additional Lipschitz hypotheses are imposed, one may obtain continuous dependence on initial data and parameters, and then study the induced maps between the corresponding solution spaces. Such continuous solution operators would interact naturally with the probability and idempotent probability functors considered above.

Several extensions are possible. The same proof scheme applies to systems with several Volterra and Fredholm terms, vector-valued kernels, and parameter-dependent equations, provided the nonlinear operators are continuous in the uniform topology. It is also natural to ask whether analogous results remain valid for delay integro-differential equations, fractional Volterra equations, equations in Banach spaces, or solution spaces endowed with compact-open rather than uniform topologies. In infinite-dimensional state spaces, compactness of bounded solution strata is no longer automatic and would require compactness or condensing assumptions on the integral operators.

10. Conclusion

We have developed a topological theory for the space of classical solutions of a nonlinear Volterra–Fredholm integro-differential equation on a compact interval. The equation combines local differential dynamics with a Volterra memory term and a Fredholm nonlocal term.

The solution space is shown to be closed in the separable Banach space $C(I, \mathbb{R}^m)$ and hence Polish. More strongly, every uniformly bounded solution stratum is compact, which implies that the full solution space is $\sigma-$compact. These structural results provide the foundation for all subsequent network and measure-theoretic conclusions.

As a consequence of second countability, the solution space has a countable network, $cs-$network, cs^*-network, $cn-$network, and $ck-$network. The space of all Borel probability measures on the solution space is again Polish and has the same countable network-type properties; the conclusion therefore goes beyond finite-support probability measures.

For each compact bounded solution stratum, both the ordinary probability-measure space and the idempotent probability-measure space are compact metrizable. Their network-type invariants are consequently countable as well. Continuous transformations between solution spaces induce the usual continuous push-forward maps on these measure spaces, giving a natural functorial interpretation of the results.

Thus, the network-type program for ODE solution spaces extends to a nonlocal integro-differential setting, but the Volterra–Fredholm structure also yields additional Polish, compactness, and measure-functor properties. These results suggest further study of delay, fractional, parameter-dependent, and infinite-dimensional integro-differential equations from the same topological viewpoint.

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