Constructing Solutions to a Fourth-Order Partial Integro-Differential Equation

A.Zh. Ashirbaeva*[1], C.B. Zholdoshova[2], G.K. Sadykova[3]
[1,2] *Osh Technological University, Osh, Kyrgyzstan*
[3] *Osh State University, Osh, Kyrgyzstan*
[1,2,3] *E-mail: aijarkyn.osh@mail.ru, chebire86@mail.ru, gsadykova@oshsu.kg*

Abstract

There is considered solution of an integro-differential equation in which the coefficients of the derivatives of the unknown function up to the fourth order inclusive are known functions. This equation is studied in an operator form, and the solution to the original problem is obtained using the method of additional argument. The prominent feature of this methodology is its ability to determine analytical solutions in some cases. Overall, the method of additional argument represents a robust contemporary framework capable of producing mathematically rigorous results.

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1. Introduction

There are many works devoted to studying solutions of first- and second-order partial differential equations. Among them, the studies by authors [21-25] stand out. At the same time, the construction of solutions for third- and higher-order equations has been investigated to a much lesser extent. Nevertheless, in recent years it has become possible to identify a number of works [1, 2, 4-12, 17-20] focused on the study of solutions to such partial differential equations. The complexity of finding solutions escalates significantly with the increasing order of the equation.

Currently, new methods are being developed to reduce the initial value problem for higher-order differential equations to integral equations. One such approach is the so-called method of an additional argument (MAA). Extensions of this method to new classes of equations were made in [3, 13, 14, 15].

The MAA method can be used to search solutions of various types of equations and conditions. Nevertheless, much of the current research focuses specifically on solving equations with given initial conditions. One of the key challenges is applying this method to boundary value problems.

Developing solutions for higher-order partial differential equations using established methods presents a more complex challenge. The problem of constructing a solution using MAA is discussed in [16]. A solution to the problem under consideration has developed, and it has proven that this solution is unique and meets the specified requirements.

This study investigates the problem of constructing solutions to integro-differential equations using the MAA, where the coefficients of the derivatives of the unknown function up to

*Corresponding author: aijarkyn.osh@mail.ru

the fourth order are given functions. It is also worth noting that the use of MAA reduces the problem to an algebraic equation. The obtained results are validated by solving a specific example.

2. Material and Methods

Formulation of the problem. The Cauchy problem for the smooth function $u(t,x)$ is considered:

$$D^2[-a(t,x)]D^2[a(t,x)]u(t,x) = f(t,x) + H(t)W[u],$$

$$W[u] := \int_0^1 \int_0^1 K(s,\xi)u(s,\xi)dsd\xi, \quad (t,x) \in G := R_+ \times R, \tag{2.1}$$

$$u(0,x) = \psi_0(x), x \in R, \tag{2.2}$$

$$\frac{\partial^k u(0,x)}{\partial t^k} = \lambda_k(x), \quad k = 1,2,3, x \in R, \tag{2.3}$$

$$a(t,x), f(t,x) \in \overline{C}^{(4)}(G), \quad \psi_0(x), \quad \lambda_k(x) \in \overline{C}^{(4)}(R), \quad (k=1,2,3), \quad H(t) \in \overline{C}^{(4)}(R_+),$$

$$K(s,\xi) \in C([0,1] \times [0,1]), \quad D[\omega] = \frac{\partial}{\partial t} + \omega\frac{\partial}{\partial x},$$

$\overline{C}^{(k)}(\Omega)$ is the space of real-valued functions being continuous and bounded with the derivatives up to the kth order.

3. The Research Findings and Discussion

Rewrite the equation (2.1) in the form:

$$D^2[-a(t,x)]D^2[a(t,x)]u(t,x) = f(t,x) + W[u] \cdot H(t). \tag{3.1}$$

For convenience, we introduce the following new functions based on conditions (2.2) and (2.3):

$$D[-a(t,x)]D^2[a(t,x)]u(t,x)\big|_{t=0} = \varphi_1(x),$$

$$D^2[a(t,x)]u(t,x)\big|_{t=0} \equiv \varphi_0(x),$$

$$D[a(t,x)]u(t,x)\big|_{t=0} = \psi_1(x),$$

These functions $p(s,t,x)$, $q(s,t,x)$ can be obtained by solving the following integral equations.

$$p(s,t,x) = x + \int_s^t a(\nu, p(\nu,t,x))d\nu, \tag{3.2}$$

$$q(s,t,x) = x - \int_s^t a(\nu, q(\nu,t,x))d\nu, \tag{3.3}$$

$$(s,t,x) \in Q := \{0 \le s \le t < \infty, x \in R\}.$$

It is essential to mention that integral equations (3.2) and (3.3) possess a unique solution based on the properties of the function under the following conditions $a(t,x) \in \overline{C}^{(4)}(G)$:

$$p(s,s,x) = x, \quad q(s,s,x) = x.$$

The integral equations (3.2) and (3.3) give the following equations:

$$D[-a(t,x)]p(s,t,x) = 0, \tag{3.4}$$

$$D[a(t,x)]q(s,t,x) = 0. \tag{3.5}$$

Lemma 3.1. The Cauchy problem (3.1), (2.2), and (2.3) can be reduced to the following integral equation, and their solutions are equivalent:

$$u(t,x) = \sum_{k=0}^{1} \psi_k(q(0,t,x))\frac{t^k}{k!} + \int_0^t (t-s)\sum_{k=0}^{1} \varphi_k(p(0,s,q(s,t,x))\frac{s^k}{k!}ds + \int_0^t (t-s)\times$$

$$\times \int_0^s (s-\tau)f(\tau, p(\tau,s,q(s,t,x)))d\tau ds + W[u]\int_0^t (t-s)\int_0^s (s-\tau)H(\tau)d\tau ds.$$

Proof. We can note: $z(t,x;u) = D^2[a(t,x)]u(t,x), \quad b(t,x) = -a(t,x)$, then (3.1) takes the following form:

$$D^2[b(t,x)]z(t,x;u) = f(t,x) + W[u]\cdot H(t). \tag{3.6}$$

if we introduce the following notation $z_1(t,x;u) = D[b(t,x)]z(t,x;u)$, then the equation (3.1) can be written as follows:

$$D[b(t,x)]z_1(t,x;u) = f(t,x) + W[u]\cdot H(t). \tag{3.7}$$

Let's use the MAA to the problem (3.7), (2.2), (2.3). As a result, we obtain the following integro-differential equation:

$$z_1(t,x;u) = \varphi_1(p(0,t,x)) + \int_0^t f(s,p(s,t,x))ds + W[u]\int_0^t H(s)ds. \tag{3.8}$$

In fact, by differentiating (3.8), we get:

$$D[b(t,x)]z_1(t,x;u) = \varphi_1'(p(0,t,x))D[b(t,x)]p(0,t,x) + f(t,x) + W[u]\cdot H(t). \tag{3.9}$$

Using equation (3.4) from (3.9), we obtain equation (3.7).
If we put $t = 0$ in equation (3.8), then $z_1(0,x;u) = \varphi_1(x)$.
Now we use the MAA for equation (3.8).

$$z(t,x;u)=\sum_{k=0}^{1}\varphi_k(p(0,t,x))\frac{t^k}{k!}+\int_0^t(t-s)f(s,p(s,t,x))ds+W[u]\int_0^t(t-s)H(s)ds. \quad (3.10)$$

Equation (3.10) satisfies (3.8) and initial conditions (2.2), (2.3).

In fact, by differentiating (3.10), we get:

$$D[b(t,x)]z(t,x;u)=z_1(t,x;u).$$

So we get (3.10) by introducing a new function $z_1(t,x;u)$. If on the contrary we use the operator $D[b(t,x)]$ for the equality of (3.10) twice in a row, it can be seen (3.1), (2.2), (2.3),

Let's introduce some more notation:

$$\theta(t,x;u)=D[a(t,x)]u(t,x),$$

Then the equation (3.10) becomes as follows:

$$D[a(t,x)]\theta(t,x;u)=$$

$$=\sum_{k=0}^{1}\varphi_k(p(0,t,x))\frac{t^k}{k!}+\int_0^t(t-s)f(s,p(s,t,x))ds+W[u]\int_0^t(t-s)H(s)ds. \quad (3.11)$$

Let's apply the MAA to the problem (3.11), (2.2), (2.3). We obtain the following equation:

$$\theta(t,x;u)=\psi_1(q(0,t,x))+\int_0^t\sum_{k=0}^{1}\varphi_k(p(0,s,q(s,t,x)))\frac{s^k}{k!}ds+$$

$$+\int_0^t\int_0^s(s-\tau)f(\tau,p(\tau,s,q(s,t,x)))d\tau ds+W[u]\int_0^t\int_0^s(s-\tau)H(\tau)d\tau ds. \quad (3.12)$$

In fact, we can prove (3.12) by differentiation:

$$D[a(t,x)]\theta(t,x;u)=\psi_1'(q)D[a(t,x)]q(0,t,x)+$$

$$+\int_0^t\sum_{k=0}^{1}\varphi_k'(p(0,s,q))\frac{\partial p}{\partial x}D[a(t,x)]q(s,t,x)ds+$$

$$+\int_0^t\int_0^s(s-\tau)\frac{\partial f}{\partial x}\frac{\partial p}{\partial x}D[a(t,x)]q(s,t,x)d\tau ds+\sum_{k=0}^{n-1}\varphi_k(p(0,t,x))\frac{t^k}{k!}+$$

$$+\int_0^t(t-s)f(s,p(s,t,x))ds+W[u]\int_0^t(t-s)H(s)ds.$$

Considering (3.5) from the last equation, there is followed equation (3.11). Also, if we put $t=0$ in equation (3.11), then $\theta(0,x;u)=\psi_1(x)$.

For problems (3.12), (2.2), (2.3), we will once again use the MAA:

$$u(t,x)=\sum_{k=0}^{1}\psi_k(q(0,t,x))\frac{t^k}{k!}+\int_0^t(t-s)\sum_{k=0}^{1}\varphi_k(p(0,s,q(s,t,x))\frac{s^k}{k!}ds+\int_0^t(t-s)\times$$

$$\times \int_0^s (s-\tau) f(\tau, p(\tau, s, q(s,t,x))d\tau ds + W[u] \int_0^t (t-s) \int_0^s (s-\tau) H(\tau) d\tau ds. \qquad (3.13)$$

In fact, by differentiating (3.13), we get:

$$D[a(t,x)]u(t,x) = [\psi_0'(q) + \psi_1'(q)t] D[a(t,x)]q(0,t,x) +$$

$$+ \int_0^t (t-s) \sum_{k=0}^{1} \varphi_k'(p(0,s,q)) \frac{\partial p}{\partial x} D[a(t,x)]q(s,t,x) +$$

$$+ \int_0^t (t-s) \int_0^s (s-\tau) \frac{\partial f}{\partial x} \frac{\partial p}{\partial x} D[a(t,x)]q(s,t,x) d\tau ds + \theta(t,x;u).$$

From the last, we get (3.12) with using the equality (3.5).
The initial condition is satisfied in (3.13): $t = 0,\ u(0,x) = \psi_0(x)$. □

Lemma 3.2. Let's

$$W_1 := \int_0^\infty \int_{-\infty}^\infty K(s,\xi) \int_0^s (s-\eta) \int_0^\eta (\eta - \tau) H(\tau) d\tau d\eta ds d\xi \neq 1.$$

The only solution to problem (2.1), (2.2), (2.3) exists in Q and is defined as follows:

$$u(t,x) = \sum_{k=0}^{1} \psi_k(q(0,t,x)) \frac{t^k}{k!} + \int_0^t (t-s) \sum_{k=0}^{1} \varphi_k(p(0,s,q(s,t,x)) \frac{s^k}{k!} ds + \int_0^t (t-s) \times$$

$$\times \int_0^s (s-\tau) f(\tau, p(\tau, s, q(s,t,x))d\tau ds + \frac{B}{1-W_1} \int_0^t (t-s) \int_0^s (s-\tau) H(\tau) d\tau ds, \qquad (3.14)$$

here

$$B = \int_0^\infty \int_0^\infty K(s,\xi) \Big\{ \sum_{k=0}^{1} \psi_k(q(0,s,\xi)) \frac{s^k}{k!} + \int_0^s (s-\tau) \sum_{k=0}^{1} \varphi_k(p(0,\tau,q(\tau,s,\xi)) \frac{\tau^k}{k!} d\tau +$$

$$+ \int_0^s (s-\tau) \int_0^\eta (\eta - \tau) f(\tau, p(\tau, \eta, q(\eta, s, \xi))) d\tau d\eta \Big\} ds d\xi.$$

Proof. It is necessary to determine the value of $W[u]$ in order for (3.13) to constitute a complete solution of the equation. For that, we multiply both sides of equation by the function $K(t,x)$ and integrate from 0 to 1 with respect to t, and from 0 to 1 with respect to x.

As a result, we get the following algebraic equation:

$$W[u] = B + W[u]W_1.$$

From this we obtain:

$$W[u] = \frac{B}{1-W_1}, \quad W_1 \neq 1.$$

Let's put (3.14) into (3.13). As a result, we get (3.13). Lemma 3.2 is proved.
Let us consider an example. □

Example. Let's $a(t,x)=1$, $H(t)=1$, $K(s,\xi)=1$ in the equation (2.1):

$$u_{tttt}(t,x)-2u_{ttxx}(t,x)+u_{xxxx}(t,x)=t+\int_0^1\int_0^1 u(s,\xi)\,d\xi\,ds. \tag{3.15}$$

Consider equation (3.15) with the following conditions:

$$u(0,x)=x^2,$$
$$u_t(0,x)=0,$$
$$u_{tt}(0,x)=x,$$
$$u_{ttt}(0,x)=0.$$

For convenience, we introduce new notation:

$$\overline{D}=\frac{\partial}{\partial t}-\frac{\partial}{\partial x},\quad D=\frac{\partial}{\partial t}+\frac{\partial}{\partial x},$$
$$\overline{D}^2D^2u(t,x)=t+\int_0^1\int_0^1 u(s,\xi)\,d\xi\,ds,$$
$$\overline{D}D^2u(t,x)\big|_{t=0}=[u_{ttt}+u_{ttx}-u_{txx}-u_{xxx}]\big|_{t=0}=1=\varphi_1(x),$$
$$D^2u(t,x)\big|_{t=0}=[u_{tt}+2u_{tx}+u_{xx}]\big|_{t=0}=x+2=\varphi_0(x),$$
$$Du(t,x)\big|_{t=0}=[u_t+u_x]\big|_{t=0}=2x=\psi_1(x),$$
$$u(0,x)=x^2=\psi_0(x).$$

As a result, we get the following solution:

$$u(t,x)=x^2+x\frac{t^2}{2}+\frac{t^5}{5!}+\frac{43}{102}\cdot\frac{t^4}{4!}.$$

4. Conclusion

The solution (3.14) of the Cauchy problem (2.1), (2.2), (2.3) for the fourth-order partial integro-differential equation, in which the coefficients of the derivatives of the unknown function up to the fourth order are known functions, has derived using the method of an additional argument.

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