

On Tightness-Type Properties of the Space of Semiadditive Functionals of Finite Support

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Abstract

We investigate tightness-type properties, in particular tightness and minitightness, of the space $OS_n(X)$ of semiadditive functionals with finite support on a compact space X . We also work with strictly τ -continuous mappings in the sense of A.V. Arkhangel'skii and show that the functor OS_n admits a natural extension to the category $Tych(\tau)$ of Tychonoff spaces and strictly τ -continuous maps. For infinite compact spaces X we prove that the functor OS_n preserves both tightness and minitightness.

Keywords: tightness, minitightness, normal functor, semiadditive functionals.

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1. Introduction

The notions of functional tightness and minitightness were introduced by A.V. Arkhangel'skii in [1] and play a central role in the study of cardinal invariants arising naturally in spaces of continuous real-valued functions. In particular, these invariants appear in the context of realcompactness and the topology of pointwise convergence on function spaces; see, for example, [1, 2, 3, 4]. The behaviour of functional tightness and minitightness under various classes of maps has been studied by several authors. In [6] O. Okunev and A.R. Paramo showed that functional tightness is not increased by R -quotient mappings and obtained bounds for functional tightness of products of locally compact spaces. In [7] Okunev analysed the minitightness of products and related it to classical cardinal invariants.

In recent years much attention has been paid to the influence of covariant functors on topological and geometric properties of various classes of spaces. For weakly normal functors acting on the category $Comp$ of compact Hausdorff spaces one can often describe the effect on cardinal invariants such as density, tightness and related quantities. For instance, in [8] R.B. Beshimov studied the (non)increase of density and weak density under weakly normal functors. In [9] the functor of semiadditive τ -smooth functionals was investigated from a categorical and cardinal point of view, while in [10] hyperspaces with a finite number of components were considered. Further results of this type for hyperspaces and measure functors can be found in [11, 12, 15], where exponential functors of finite degree and the functor of probability measures of finite support are shown to preserve various cardinal invariants, including functional tightness and, in some cases, local density.

A separate and very fruitful direction is based on functors generated by order-preserving or semiadditive functionals on $C(X)$. In [18] T. Radul introduced the functor OS of weakly additive, order-preserving and normed functionals and established several fundamental properties of this construction. Structural and geometric aspects of the compacta $OS(X)$ were

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later analysed in [13, 14], where the extreme boundary of the compacta $OS(n)$ and $OS(3)$ was described.

From the categorical side we use the standard extension F_β of a normal functor $F: \mathbf{Comp} \rightarrow \mathbf{Comp}$ to the category $\mathbf{Tych}$ as defined in [20], together with the Basmanov representation of $F_n(X)$ as a continuous image of the product $X^n \times F(n)$ [21]. In particular, for the functor OS of semiadditive functionals the Basmanov mapping

$$\pi_{OS,X,n}: X^n \times OS(n) \longrightarrow OS_n(X)$$

is a perfect (hence closed and onto) map. This representation will be crucial in our analysis of tightness and minitightness for $OS_n(X)$.

On the level of cardinal invariants we rely on classical results on tightness and minitightness in products. Malykhin proved in [22] that tightness behaves well under finite products of compact spaces, while Krupski showed in [19] that minitightness of an infinite product can be controlled in terms of the minitightness of the factors. Combining these results with the Basmanov representation and the fact that $OS(n)$ is a separable metric space, we obtain precise equalities for the tightness and minitightness of $OS_n(X)$ in terms of the corresponding invariants of X .

The main results of the paper can be summarised as follows. For every infinite compact space X and every natural number n we prove that

$$t(OS_n(X)) = t(X) \quad \text{and} \quad t_m(OS_n(X)) = t_m(X).$$

Moreover, for arbitrary spaces we show that

$$t_m(X) \leq t_m(OS_n(X)),$$

so that OS_n does not decrease minitightness. In addition, we describe the behaviour of strictly τ -continuous real-valued functions under the functor OS_n and relate our results to realcompactness, using the classical characterisation of realcompact function spaces due to Arkhangel'skii and Chigogidze [1].

The paper is organized as follows. In Section 2 we collect basic definitions and facts about normal functors, supports and semiadditive functionals that will be used later; for general topology we refer to [16, 17]. In Section 3 we introduce the categories $\mathbf{Comp}(\tau)$ and $\mathbf{Tych}(\tau)$ and construct an extension of the functor OS_n over strictly τ -continuous mappings. Section 4 is devoted to tightness and minitightness: we prove that OS_n preserves tightness and minitightness of infinite compact spaces and discuss the connection with realcompactness of function spaces.

2. Preliminary Information

Definition 2.1. A functor $F: \mathbf{Comp} \rightarrow \mathbf{Comp}$ is called:

- 1) weight-preserving if $\omega(F(X)) = \omega(X)$ for every compact space X ;
- 2) monomorphic if for any embedding i of a compact X into a compact Y , the map $F(i): F(X) \rightarrow F(Y)$ is also an embedding;
- 3) epimorphic if it preserves surjective maps of compacta;

- 4) intersection-preserving if for any family $\{X_\alpha : \alpha \in \Omega\}$ of closed subsets of an arbitrary compact X we have

$$\bigcap_{\alpha \in \Omega} F(X_\alpha) = F\left(\bigcap_{\alpha \in \Omega} X_\alpha\right);$$

- 5) preimage-preserving if for any continuous map f of a compact Y and for every closed subset $B \subseteq Y$ we have $F(f^{-1}B) = F(f)^{-1}F(B)$.

A covariant functor $F: \text{Comp} \rightarrow \text{Comp}$ is called normal if it is continuous, preserves weight, intersections and preimages, is monomorphic and epimorphic, and transfers singletons into singletons and the empty set into the empty set.

A closed subset $\text{supp}_F(x) \subseteq X$ such that the relations $\text{supp}_F(x) \subseteq A$ and $x \in F(A)$ are equivalent for any closed set $A \subseteq X$ is called the support of the point $x \in F(X)$.

Let $F: \text{Comp} \rightarrow \text{Comp}$ be an arbitrary normal functor and $X \in \text{Ob}(\text{Tych})$. Assume that

$$F_\beta(X) = \{x \in F(\beta X) : \text{supp}(x) \subseteq X\}.$$

If $f: X \rightarrow Y$ is a morphism of the category Tych , then we put $F_\beta(f) = F(\beta f)|_{F_\beta(X)}$, where $\beta f: \beta X \rightarrow \beta Y$ is the Stone-Cech compactification of the map f . Using the normality of the functor F , we can show that the definition of $F_\beta(f)$ is correct, that is $F(\beta f)(F_\beta(X)) \subseteq F_\beta(Y)$. It is also easy to see that F_β is a covariant functor of the category Tych into itself [20].

Let X be a compact space. By $C(X)$ we denote the set of all continuous functions $\varphi: X \rightarrow \mathbb{R}$ with the usual algebraic operations. There is a sup-norm on this set defined as follows:

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X\}.$$

A constant function is denoted by c_X and defined by the equality

$$c_X(x) = c,$$

for all $x \in X$ and for each real number c .

Let φ, ψ be elements of $C(X)$. We write $\varphi \leq \psi$ if $\varphi(x) \leq \psi(x)$ for all $x \in X$.

A mapping $\nu: C(X) \rightarrow \mathbb{R}$ is called a functional.

Definition 2.2 ([18]). A functional $\mu: C(X) \rightarrow \mathbb{R}$ is called:

- 1) weakly additive if $\mu(\varphi + c_X) = \mu(\varphi) + c$ for any $\varphi \in C(X)$ and $c \in \mathbb{R}$;
- 2) order-preserving if the inequality $\varphi \leq \psi$ implies $\mu(\varphi) \leq \mu(\psi)$ for any pair of functions $\varphi, \psi \in C(X)$;
- 3) normed if $\mu(1_X) = 1$;
- 4) positively homogeneous if $\mu(\lambda\varphi) = \lambda\mu(\varphi)$ for all $\varphi \in C(X)$ and $\lambda \in \mathbb{R}_+ = [0, +\infty)$;
- 5) semiadditive if $\mu(f + g) \leq \mu(f) + \mu(g)$ for all $f, g \in C(X)$.

We denote the set of all weakly additive, order-preserving, normed, positively homogeneous and semiadditive functionals on a compact space X by $OS(X)$. For the sake of brevity, we call the elements of the set $OS(X)$ semiadditive functionals. Note that every functional $\nu \in OS(X)$ is a continuous map from $C(X)$ to $\mathbb{R}$ (see [18]). Hence, the set $OS(X)$ is a subset of the space $C_p(C(X))$ of all continuous functions on $C(X)$ provided with the topology of

pointwise convergence. The neighbourhood base of a semiadditive functional $\nu \in OS(X)$ is formed by sets of the form:

$$(\nu; \varphi_1, \dots, \varphi_k; \varepsilon) = \{\nu' \in OS(X) : |\nu(\varphi_i) - \nu'(\varphi_i)| < \varepsilon, i = 1, 2, \dots, k\},$$

where $\varphi_i \in C(X)$, $i = 1, 2, \dots, k$, and $\varepsilon > 0$.

Let $X, Y \in Comp$ and $f: X \rightarrow Y$ be a continuous map. Define the mapping $OS(f): OS(X) \rightarrow OS(Y)$ using the formula

$$(OS(f)(\mu))(\varphi) = \mu(\varphi \circ f),$$

where $\mu \in OS(X)$ and $\varphi \in C(Y)$.

In [5], the authors proved that the covariant functor $OS: Comp \rightarrow Comp$ of semiadditive functionals is normal.

3. On an Extension of the Functor OS_n over Strictly τ -Continuous Mappings

Recall that a functional μ is said to be supported on a set F if $\varphi|_F = 0$ implies $\mu(\varphi) = 0$ for every $\varphi \in C(X)$. The minimal closed set F on which the functional μ is supported is called the support of μ . The support of a functional μ is denoted by $supp(\mu)$.

For a Tychonoff space X let $C_b(X)$ be the algebra of all bounded continuous real-valued functions with the pointwise algebraic operations. For a function $\varphi \in C_b(X)$ put

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X\}.$$

By βX we denote the Stone-Cech compactification of the space X . For each function $\varphi \in C_b(X)$ consider its continuous extension $\tilde{\varphi} \in C(\beta X)$. This gives an isomorphism between the spaces $C_b(X)$ and $C(\beta X)$; moreover, $\|\tilde{\varphi}\| = \|\varphi\|$, i.e., this isomorphism is an isometry, and the topological properties of these spaces coincide. For this reason, one may consider any function from $C_b(X)$ as an element of $C(\beta X)$.

For a space X and the functor OS put

$$OS_n(X) = \{\mu \in OS_\beta(X) : |supp(\mu)| \leq n\},$$

where n is a natural number, and

$$OS_\omega(X) = \{\mu \in OS_\beta(X) : |supp(\mu)| < \infty\}.$$

Clearly,

$$OS_\omega(X) = \bigcup_{n=1}^{\infty} OS_n(X).$$

For a finite discrete space $X = \{1, 2, \dots, n\}$ and a weakly normal functor F put $F(n) = F(X)$.

Recall the V.N. Basmanov mapping $\pi_{F,X,n}: X^n \times F(n) \rightarrow F_n(X)$, where $F: Comp \rightarrow Comp$ is a normal functor acting in the category of compacta and n is a natural number. We identify X^n with the set $C(n, X)$ of mappings from $\{1, 2, \dots, n\}$ into X and put $\pi_{F,X,n}(\xi, a) = F(\xi)(a)$, where $\xi \in X^n$, $a \in F(n)$. Note that the mapping $\pi_{F,X,n}$ is continuous [21].

Definition 3.1. A continuous mapping $f: X \rightarrow Y$ is called perfect if f is closed and every fiber $f^{-1}(y)$ is a compact subset of X .

Note that for the functor $OS: Comp \rightarrow Comp$ of semiadditive functionals the V.N. Basmanov mapping

$$\pi_{OS,X,n}: X^n \times OS(n) \rightarrow OS_n(X)$$

is a perfect map (in particular, closed and onto).

Remark 3.1. For an arbitrary Tychonoff space X let us define a mapping $\delta_X: X \rightarrow OS_n(X)$ which assigns to each point $x \in X$ the Dirac measure $\delta_X(x) = \delta_x$ concentrated at the point x .

Note that every mapping $\delta_X: X \rightarrow OS_n(X)$ is a closed embedding.

Definition 3.2 ([1]). Let X and Y be topological spaces. A function $f: X \rightarrow Y$ is said to be strictly τ -continuous if for every subspace A of X such that $|A| \leq \tau$, the restriction of f to A coincides with the restriction to A of some continuous function $g: X \rightarrow Y$.

Let us denote by $Comp(\tau)$ (respectively, $Tych(\tau)$) the category of compact (Tychonoff) spaces and their strictly τ -continuous mappings. Our aim is to construct an extension of the functor OS_n over the category $Tych(\tau)$.

Proposition 3.1. The functor $OS_n: Tych \rightarrow Tych$ can be extended to the category $Tych(\tau)$ for every infinite cardinal number τ .

Proof. Let $\nu \in OS_n(X)$ and $supp(\nu) = \{x_1, x_2, \dots, x_n\}$. For a strictly τ -continuous function $f: X \rightarrow \mathbb{R}$ there exists a continuous function $\varphi \in C_b(X)$ such that $\varphi(x_i) = f(x_i)$ for $i = 1, 2, \dots, n$. We put $\nu(f) = \nu(\varphi)$. Clearly, the value of $\nu(f)$ does not depend on the choice of a continuous function $\varphi \in C_b(X)$. Indeed, for every continuous function $\psi \in C_b(X)$ such that $\psi(x_i) = f(x_i)$ ($i = 1, 2, \dots, n$), we have $\nu(\varphi) = \nu(\psi)$. With the above construction, we have extended a semiadditive functional of finite support to the class of all strictly τ -continuous functions. This extension is also weakly additive, i.e., the following conditions hold:

- 1) $\nu(c_X + f) = c + \nu(f)$;
- 2) $\nu(f) \leq \nu(g)$ whenever $f \leq g$

for a constant $c \in \mathbb{R}$ and strictly τ -continuous functions f, g .

Indeed, for strictly τ -continuous functions f and g there exist continuous functions φ_f and φ_g such that $\nu(f) = \nu(\varphi_f)$ and $\nu(g) = \nu(\varphi_g)$. Clearly, the sum of a constant function and a strictly τ -continuous function is again strictly τ -continuous, and we have

$$\nu(c_X + f) = \nu(c_X + \varphi_f) = c + \nu(\varphi_f) = c + \nu(f).$$

Suppose now $f \leq g$. Then $\varphi_f(x_i) \leq \varphi_g(x_i)$ for $i = 1, 2, \dots, n$, and consequently, we obtain

$$\nu(f) = \nu(\varphi_f) \leq \nu(\varphi_g) = \nu(g).$$

Now consider a strictly τ -continuous mapping $h: X \rightarrow Y$ between Tychonoff spaces X and Y . Define the mapping $OS_n(h): OS_n(X) \rightarrow OS_n(Y)$ by the formula

$$(OS_n(h)(\mu))(\varphi) = \mu(\varphi \circ h),$$

for $\mu \in OS_n(X)$ and $\varphi \in C_b(Y)$. We have to show that the mapping $OS_n(h): OS_n(X) \rightarrow OS_n(Y)$ is strictly τ -continuous.

Suppose $\Phi \subset OS_n(X)$ with $|\Phi| \leq \tau$. Put

$$E = \bigcup \{supp(\mu_\alpha) : \mu_\alpha \in \Phi\}.$$

Then clearly $|E| \leq \tau$. There exists a continuous mapping $u: X \rightarrow Y$ such that $u|_E = h|_E$. It is known that $OS_n(u): OS_n(X) \rightarrow OS_n(Y)$ is continuous. Besides, we have

$$(OS_n(u)(\mu_\alpha))(\varphi) = \mu_\alpha(\varphi \circ u) = \mu_\alpha(\varphi \circ h) = (OS_n(h)(\mu_\alpha))(\varphi)$$

since $\varphi \circ u|_E = \varphi \circ h|_E$ and $supp(\mu_\alpha) \subset E$. Thus we have proved that $OS_n(h): OS_n(X) \rightarrow OS_n(Y)$ is strictly τ -continuous. Now we have a functor $OS_n: Tych(\tau) \rightarrow Tych(\tau)$ acting in the category $Tych(\tau)$ of Tychonoff spaces and their strictly τ -continuous mappings. Proposition 3.1 is proved. $\square$

Problem 3.1. Extend the functor $OS: Comp \rightarrow Comp$ over the category $Comp(\tau)$ for some cardinal number τ .

Note that if $f: X \rightarrow Y$ is strictly τ -continuous, then the restriction $f|_A: A \rightarrow Y$ of f to any subspace $A \subset X$ is also strictly τ -continuous.

For a function $f: X \rightarrow \mathbb{R}$ let us define a function $\Phi_f: OS_n(X) \rightarrow \mathbb{R}$ with the following formula:

$$\Phi_f(\nu) = \nu(f)$$

for $\nu \in OS_n(X)$.

The function $\Phi_f: OS_n(X) \rightarrow \mathbb{R}$ is continuous for any continuous function $f: X \rightarrow \mathbb{R}$.

Proof. Suppose $f: X \rightarrow \mathbb{R}$ is a continuous function. Take an arbitrary functional $\nu_0 \in OS_n(X)$. Then $\Phi_f(\nu_0) = \nu_0(f) \in \mathbb{R}$. Consider an arbitrary ε -neighbourhood $U_\varepsilon(\nu_0(f)) = (\nu_0(f) - \varepsilon, \nu_0(f) + \varepsilon)$ of $\nu_0(f)$ in $\mathbb{R}$. Let $supp(\nu_0) = \{x_1, x_2, \dots, x_n\}$. Now we consider the neighbourhood $O(\nu_0; f; \varepsilon)$ of ν_0 in $OS_n(X)$ and show that $\Phi_f(O(\nu_0; f; \varepsilon)) \subset U_\varepsilon(\nu_0(f))$. Indeed, take an arbitrary element $\nu \in O(\nu_0; f; \varepsilon)$. Then we have

$$|\Phi_f(\nu) - \Phi_f(\nu_0)| = |\nu(f) - \nu_0(f)| < \varepsilon,$$

which means that $\Phi_f(\nu) \in U_\varepsilon(\nu_0(f))$. This proves that $\Phi_f: OS_n(X) \rightarrow \mathbb{R}$ is continuous. Lemma 3 is proved. $\square$

Proposition 3.2. The function $\Phi_f: OS_n(X) \rightarrow \mathbb{R}$ is strictly τ -continuous for any strictly τ -continuous function $f: X \rightarrow \mathbb{R}$.

Proof. Let $f: X \rightarrow \mathbb{R}$ be a strictly τ -continuous function. We have to show that the function $\Phi_f: OS_n(X) \rightarrow \mathbb{R}$ is strictly τ -continuous. Let $\Omega \subset OS_n(X)$ with $|\Omega| \leq \tau$. Put

$$M = \bigcup \{supp(\mu) : \mu \in \Omega\}.$$

Then clearly $|M| \leq \tau$. There exists a continuous function $g: X \rightarrow \mathbb{R}$ such that $g|_M = f|_M$, since f is strictly τ -continuous. Now by Lemma 3 the function $\Phi_g: OS_n(X) \rightarrow \mathbb{R}$ is continuous. Moreover, since $g|_M = f|_M$ implies $\mu(g) = \mu(f)$ for every $\mu \in \Omega$, we obtain

$$\Phi_g(\mu) = \mu(g) = \mu(f) = \Phi_f(\mu),$$

and therefore, $\Phi_g|_\Omega = \Phi_f|_\Omega$. This proves that $\Phi_f: OS_n(X) \rightarrow \mathbb{R}$ is strictly τ -continuous. Proposition 4.2 is proved. $\square$

4. Tightness-Type Properties of the Space $OS_n(X)$

Definition 4.1. [[1]] Let X and Y be topological spaces. A function $f: X \rightarrow Y$ is said to be τ -continuous if for every subspace A of X such that $|A| \leq \tau$, the restriction $f|_A$ is continuous.

Proposition 4.1 ([6]). A mapping $f: X \rightarrow Y$ is τ -continuous if and only if for every closed set F in Y , the preimage $f^{-1}(F)$ is τ -closed in X .

Recall that a subset F of a space X is τ -closed if for every $B \subset F$ with $|B| \leq \tau$, the closure in X of the set B is contained in F . The τ -closure of a set A is defined as

$$[A]_\tau = \bigcup \{\overline{B} : B \subset A, |B| \leq \tau\}.$$

A set A is said to be τ -dense in X if $[A]_\tau = X$.

The following result generalizes Proposition 4.1.

Proposition 4.2 ([11]). For a mapping $f: X \rightarrow Y$ of arbitrary topological spaces X and Y the following conditions are equivalent:

- 1) the mapping $f: X \rightarrow Y$ is τ -continuous;
- 2) for every closed set F in Y , the preimage $f^{-1}(F)$ is τ -closed in X ;
- 3) for every τ -closed set F in Y , the preimage $f^{-1}(F)$ is τ -closed in X ;
- 4) $f([A]_\tau) \subset [f(A)]_\tau$ for an arbitrary subset $A \subset X$;
- 5) $[f^{-1}(B)]_\tau \subset f^{-1}([B]_\tau)$ for an arbitrary subset $B \subset Y$.

Note that from Proposition 4.2 we directly have the following.

Corollary 4.1. If $f: X \rightarrow Y$ is a continuous function, then the preimage of every τ -closed set is τ -closed.

The tightness $t(X)$ of a topological space X is the smallest infinite cardinal number τ such that the following condition is satisfied: if $x \in X$, $A \subset X$ and $x \in \overline{A}$, then there exists a set $B \subset A$ for which $|B| \leq \tau$ and $x \in \overline{B}$.

Note that the tightness $t(X)$ is less or equal to τ if and only if every τ -closed set in X is closed.

Let $f: X \rightarrow Y$ be a closed mapping of a space X onto a space Y . Then $t(Y) \leq t(X)$.

Proof. Suppose $t(X) \leq \tau$. Take an arbitrary τ -closed subset $A \subset Y$ and an arbitrary point $y \in Y \setminus A$. Then clearly $f^{-1}(y) \cap f^{-1}(A) = \emptyset$. By virtue of Corollary 4.1, $f^{-1}(A)$ is τ -closed in X . Since $t(X) \leq \tau$, we obtain that $f^{-1}(A)$ is closed. Then there exists an open subset $U \subset X$ such that $f^{-1}(y) \subset U$ and $U \cap f^{-1}(A) = \emptyset$. Now, by Theorem 1.4.13 of [17] there exists a neighbourhood V of y in Y such that $f^{-1}(V) \subset U$. Therefore, $f^{-1}(V) \cap f^{-1}(A) = \emptyset$, which implies $V \cap A = \emptyset$, since f is surjective. Thus, $Y \setminus A$ is open, and consequently, A is closed. This shows that $t(Y) \leq \tau$. Lemma 4 is proved. $\square$

Theorem 4.1. For an arbitrary infinite compact space X we have

$$t(X) = t(OS_n(X)).$$

Proof. Let us first show the inequality $t(X) \leq t(OS_n(X))$. Note that by Remark 3.1 the space X can be embedded in $OS_n(X)$ as a closed subspace. Since tightness is monotone with respect to arbitrary subspaces, we obtain that $t(X) \leq t(OS_n(X))$.

Let us now prove the inequality $t(OS_n(X)) \leq t(X)$. Suppose $t(X) = \tau$, where τ is some infinite cardinal number. By Theorem 4 of [22] we have $t(X^n) = \tau$. Besides, since the space $OS(n)$ is a separable metric space, the tightness of $OS(n)$ is countable. Now again by Theorem 4 of [22] we obtain $t(X^n \times OS(n)) = \tau$. Since V.N. Basmanov's mapping

$$\pi_{OS,X,n}: X^n \times OS(n) \rightarrow OS_n(X)$$

is closed and onto, by Lemma 4 we have that $t(OS_n(X)) = \tau$. Theorem 4.1 is proved. $\square$

In [1] A.V. Arkhangel'skii introduced the minitightness of a topological space as follows.

Definition 4.2. The minitightness of a space X is

$$t_m(X) = \min\{\tau : \tau \text{ is an infinite cardinal and every strictly } \tau\text{-continuous real-valued function on } X \text{ is continuous}\}.$$

Theorem 4.2. For every infinite space X the following inequality holds:

$$t_m(X) \leq t_m(OS_n(X)).$$

Proof. Let $t_m(OS_n(X)) = \tau$. We are going to prove that an arbitrary strictly τ -continuous function $f: X \rightarrow \mathbb{R}$ is continuous. Indeed, by Proposition 4.2 the mapping $\Phi_f: OS_n(X) \rightarrow \mathbb{R}$ is also strictly τ -continuous. By the equality $t_m(OS_n(X)) = \tau$ we obtain that the mapping $\Phi_f: OS_n(X) \rightarrow \mathbb{R}$ is continuous.

Since Φ_f is continuous on the subspace $\delta(X)$ (see Remark 3.1), we obtain continuity of $f: X \rightarrow \mathbb{R}$ from the fact that $\Phi_f(\delta_x) = \delta_x(f) = f(x)$ for every $x \in X$. Theorem 4.2 is proved. $\square$

Corollary 4.2. For every infinite compact space X we have

$$t_m(X) = t_m(OS_n(X)).$$

Proof. By Theorem 4.2 we already have the inequality $t_m(X) \leq t_m(OS_n(X))$. It remains to show that $t_m(OS_n(X)) \leq t_m(X)$ for an arbitrary compact X . By Corollary 3.3 in [19] we have $t_m(X) = t_m(X^n)$ for any natural number n . Since a quotient mapping does not increase the minitightness of topological spaces (see Proposition 3 in [1]) and V.N. Basmanov's mapping

$$\pi_{OS,X,n}: X^n \times OS(n) \rightarrow OS_n(X)$$

is quotient, we obtain the inequality $t_m(OS_n(X)) \leq t_m(X)$. Corollary 4.2 is proved. $\square$

Definition 4.3 ([17]). A topological space X is called realcompact if X is a Tychonoff space and there is no Tychonoff space $\tilde{X}$ which satisfies the following two conditions:

- 1) there exists a homeomorphic embedding $r: X \rightarrow \tilde{X}$ such that $r(X) \neq \overline{r(X)} = \tilde{X}$;
- 2) for every continuous real-valued function $f: X \rightarrow \mathbb{R}$ there exists a continuous function $\tilde{f}: \tilde{X} \rightarrow \mathbb{R}$ such that $\tilde{f} \circ r = f$.

The question of when $C_p(X)$, the space of all real-valued continuous functions defined on X with the topology of pointwise convergence, is realcompact, arouses interest in general topology. A sufficient and necessary condition was found by A.V. Arkhangel'skii and A.Ch. Chigogidze, respectively: X is normal and every ω -continuous function $f: X \rightarrow Y$ is continuous. Finally, using a slight modification of the concept of τ -continuity, A.V. Arkhangel'skii gave a complete answer to the posed question.

Theorem 4.3 ([1]). *For a Tychonoff space X the following conditions are equivalent:*

- 1) $C_p(X)$ is realcompact;
- 2) every strictly τ -continuous function $f: X \rightarrow Y$ is continuous.

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