

On Baire Property of Spaces of Measurable Functions

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Abstract

A Tychonoff space X is called a Baire space if the intersection of any countable family of dense open sets in X is again dense in X . One interesting problem in the theory of function spaces is to study when such a function space becomes a Baire space, particularly, in terms of the support of functions considered.

In this paper, it is shown that $M(X, K)$ of all measurable K -valued functions defined on a measurable space (X, Σ) is a Baire space if K is a compact metric space and $M(X, K)$ is equipped with the topology of pointwise convergence.

Keywords: Baire property, function space, measurable function

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1. Introduction

A space is *meager* (or *of the first Baire category*) if it can be written as a countable union of closed sets with empty interior. A topological space X is *Baire* if the Baire Category Theorem holds for X , i.e., the intersection of any sequence of open dense subsets of X is dense in X . Clearly, if X is Baire, then X is not meager. The reverse implication is in general not true. However, it holds for every homogeneous Tychonoff space X (see Theorem 2.3 in [3]).

The Baire property is an important topological property used not only in topology but also in other areas of mathematics, such as category theory, functional analysis, measure theory and others, and so it is natural to explore the question: *When does a function space have (or not have) the Baire property?*

The Baire property for the space of continuous mappings was first considered in [3] and [2]. In [3], necessary and, in some cases, sufficient conditions on the space X were obtained under which the space $C_p(X, \mathbb{R})$ of all continuous real-valued functions defined on the space X with the topology p of pointwise convergence, has the Baire property.

In general, characterizing the Baire property for function spaces is not a simple task. For example, the problem for the space $C_p(X, \mathbb{R})$ remained unsolved for a long time. Later, the problem for $C_p(X, \mathbb{R})$ was solved independently by E.G. Pytkeev in [11], V. Tkachuk in [12] and van Douwen in [4]. For the space $C_p(X, [0, 1])$, the Baire property was studied in [9]. For the space $B_1(X, \mathbb{R})$ of Baire-one functions the problem was solved in [7].

2. Main Definitions and Notation

The set of positive integers is denoted by $\mathbb{N}$ and $\omega = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}$ be the real line, we put $\mathbb{I} = [0, 1] \subset \mathbb{R}$, and let $\mathbb{Q}$ denote the set of rational numbers.

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Let $V = V[x_1, \dots, x_n, p_1, \dots, p_n] = \{f \in \{0, 1\}^X : f(x_i) \in p_i, i = 1, \dots, n\}$ where $x_i \in X$, $p_i \in \{0, 1\}$ for $i = 1, \dots, n$. Denote by $\text{supp}V := \{x_1, \dots, x_n\}$.

The basic open neighborhood of a function $f \in \{0, 1\}^X$ with support A will be the set $O(f, A) := \{h \in \{0, 1\}^X : h \upharpoonright A = f \upharpoonright A\}$, where A is a finite subset of the space X . Note that $\text{supp}O(f, A) = A$.

Recall that (X, Σ) is a *measurable space*, if X is a set equipped with σ -algebra Σ . In mathematical analysis and in probability theory, a σ -algebra on a set X is a nonempty collection Σ of subsets of X closed under complement, countable unions, and countable intersections.

Definition 2.1. Let (X, Σ) be a measurable space and $A, B \subset X$. Then A and B are Σ -separated if there is $C \in \Sigma$ such that $A \subseteq C$ and $B \subseteq X \setminus C$.

Let (X, Σ) be a measurable space and let Y be a topological space. Then, a function $f : X \rightarrow Y$ is said to be *measurable* if $f^{-1}(V) \in \Sigma$ for every open subset V of Y .

A function $f : X \rightarrow \{0, 1\}$ is said to be *indicator measurable* if for every $i \in \{0, 1\}$ the pre-image $f^{-1}(i) \in \Sigma$.

Denote by $\mathbf{2}$ the doubleton, i.e., $\mathbf{2} = \{0, 1\}$ endowed with the discrete topology. Denote by $M(X, \mathbf{2})$ the set of all indicator measurable functions with the pointwise convergence topology. If Y is a topological space then denote by $M(X, Y)$ the set of all measurable functions with the pointwise convergence topology.

Note that the following proposition holds.

Definition 2.2. Let (X, Σ) be a measurable space. Then $M(X, \mathbf{2})$ is dense in $\mathbf{2}^X$ if and only if for any points $x, y \in X$ ($x \neq y$) the sets $\{x\}$ and $\{y\}$ are Σ -separated.

Further, we assume that σ -algebra Σ has the property that for any points $x, y \in X$ ($x \neq y$) the sets $\{x\}$ and $\{y\}$ are Σ -separated.

For other notation almost without exceptions we follow Engelking's book [5].

Definition 2.3. We say that a measurable space (X, Σ) has the M -property if every pairwise disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of finite subsets $\Delta_n \subset X$, $\Delta_n = A_n \cup B_n$, $A_n \cap B_n = \emptyset$, $n \in \mathbb{N}$, contains subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$ such that the sets $\bigcup_{k=1}^{\infty} A_{n_k}$ and $\bigcup_{k=1}^{\infty} B_{n_k}$ are Σ -separated.

The following two lemmas were proved for the space $B(X, \mathbf{2})$ of Baire indicator functions in [6]. The proofs of these lemmas for the space $M(X, \mathbf{2})$ are repeated by formally replacing the space $B(X, \mathbf{2})$ with the space $M(X, \mathbf{2})$. For the convenience of the reader, we present these proofs.

Lemma 2.1. Let (X, Σ) be a measurable space. Then the following statements are equivalent:

1. X has the M -property;
2. for any pairwise disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of finite subsets $\Delta_n \subset X$ and a function $f : \bigcup_{n=1}^{\infty} \Delta_n \rightarrow \{0, 1\}$ there is a subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$ and $\tilde{f} \in M(X, \mathbf{2})$ such that $\tilde{f} \upharpoonright \bigcup_{k=1}^{\infty} \Delta_{n_k} = f \upharpoonright \bigcup_{k=1}^{\infty} \Delta_{n_k}$.

Proof. (1) $\Rightarrow$ (2). Let $\{\Delta_n : n \in \mathbb{N}\}$ be a pairwise disjoint sequence of finite subsets of X , $f : \bigcup_{n=1}^{\infty} \Delta_n \rightarrow \{0, 1\}$. Let $A_n = f^{-1}(0) \cap \Delta_n$, $B_n = f^{-1}(1) \cap \Delta_n$ for every $n \in \mathbb{N}$. By (1), there is a subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$ such that the sets $A = \bigcup_{k=1}^{\infty} A_{n_k}$ and $B = \bigcup_{k=1}^{\infty} B_{n_k}$ are Σ -separated, i.e. there is $C \in \Sigma$ such that $A \subseteq C$ and $B \subseteq X \setminus C$. Consider the function $\tilde{f} \in M(X, \mathbf{2})$ such that $\tilde{f}(C) \subseteq \{0\}$ and $\tilde{f}(X \setminus C) \subseteq \{1\}$. It is clear that $\tilde{f} \upharpoonright \bigcup_{k=1}^{\infty} \Delta_{n_k} = f \upharpoonright \bigcup_{k=1}^{\infty} \Delta_{n_k}$.

(2) $\Rightarrow$ (1). Let $\{\Delta_n : n \in \mathbb{N}\}$ be a pairwise disjoint sequence of finite subsets of X , $\Delta_n = A_n \cup B_n$, $A_n \cap B_n = \emptyset$, $n \in \mathbb{N}$. Let $f : \bigcup_{n=1}^{\infty} \Delta_n \rightarrow \{0, 1\}$ such that $f(x) = 0$ for $x \in \bigcup_{n=1}^{\infty} A_n$ and $f(x) = 1$ for $x \in \bigcup_{n=1}^{\infty} B_n$. By (2), there is a subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$ and $\tilde{f} \in M(X, \mathbf{2})$ such that $\tilde{f} \upharpoonright \bigcup_{k=1}^{\infty} \Delta_{n_k} = f \upharpoonright \bigcup_{k=1}^{\infty} \Delta_{n_k}$. It is clear that the subsequence $\{\Delta_{n_k} : k \in \mathbb{N}\}$ required where the sets $A = \bigcup_{k=1}^{\infty} A_{n_k}$ and $B = \bigcup_{k=1}^{\infty} B_{n_k}$ are Σ -separated by $C = \tilde{f}^{-1}(0)$. $\square$

Lemma 2.2. Let (X, Σ) be a measurable space, $\Delta = \{x_1, \dots, x_m\}$ be a finite subset of X and F be a closed nowhere dense subset of $M(X, \mathbf{2})$. Then there is an base open set V in the space $M(X, \mathbf{2})$ such that

- (a) $\text{supp}V \cap \Delta = \emptyset$;
- (b) for any $f \in V$ there is $\varphi : A \rightarrow \{0, 1\}$ where $A := \Delta \cup \text{supp}V$ such that $\varphi = f \upharpoonright A$ and $\varphi \notin \overline{\{h \upharpoonright A : h \in F\}}^{2^A}$.

Proof. Renumber all points of the set 2^Δ as $\{t_i : 1 \leq i \leq 2^\Delta\}$. Denote by $W(\Delta, t_i) := \{f \in M(X, \mathbf{2}) : f \upharpoonright \Delta = t_i\}$ for every $1 \leq i \leq 2^\Delta$.

Construct by induction base open sets W_i, V_i , $1 \leq i \leq 2^\Delta$ such that

- (1) $\text{supp}W_i = \Delta$ for every $1 \leq i \leq 2^\Delta$;
- (2) $V_{i+1} \subseteq V_i$, $\text{supp}V_{i+1} \supseteq \text{supp}V_i$ for every $1 \leq i \leq 2^\Delta - 1$;
- (3) $\text{supp}V_i \cap \Delta = \emptyset$;
- (4) $W_{i+1} \cap V_{i+1} \subseteq (W(\Delta, t_{i+1}) \cap V_i) \setminus F$.

Since $W(\Delta, t_1)$ is open and nonempty, there are base open sets W_1 and V_1 such that $W_1 \cap V_1 \subseteq W(\Delta, t_1) \setminus F$, $\text{supp}W_1 = \Delta$, $\text{supp}V_1 \cap \Delta = \emptyset$.

Let the sets W_i, V_i , $1 \leq i \leq k < 2^\Delta$ be constructed.

Then the set $W(\Delta, t_{k+1}) \cap V_k$ is open and non-empty. Hence, there are base open sets W_{k+1}, V_{k+1} such that $W_{k+1} \cap V_{k+1} \subseteq (W(\Delta, t_{k+1}) \cap V_k) \setminus F$, $\text{supp}W_{k+1} = \Delta$, $\text{supp}V_{k+1} \supseteq \text{supp}V_k$, $V_{k+1} \subseteq V_k$.

Thus, the sets W_i, V_i , $1 \leq i \leq 2^\Delta$ are constructed.

We show that $V = V_{2^\Delta}$ is required. Note that $\text{supp}V \cap \Delta = \emptyset$. Let $f \in V$. Then there is i_0 such that $t_{i_0} = f \upharpoonright \Delta$.

Let $W_{i_0} = \{g \in M(X, \mathbf{2}) : g(P) \subseteq \{0\}, g(Q) \subseteq \{1\}\}$ for finite subsets P and Q of X such that $P \cap Q = \emptyset$ and $P \cup Q = \Delta$.

Let $A := \Delta \cup \text{supp}V$. Consider the function $\varphi : A \rightarrow \{0, 1\}$ such that $\varphi(P) \subseteq \{0\}$, $\varphi(Q) \subseteq \{1\}$, $\varphi \upharpoonright \text{supp}V = f \upharpoonright \text{supp}V$.

Since $W_{i_0} \cap V_{i_0} \subseteq W(\Delta, t_{i_0})$ and $V \subseteq V_{i_0}$, $W_{i_0} \cap V \subseteq W(\Delta, t_{i_0})$. Note that $\varphi = f \upharpoonright A$.

Since $(W_{i_0} \cap V) \cap F = \emptyset$, $\varphi \notin \overline{\{h \upharpoonright A : h \in F\}}^{2^A}$. □

3. Main results

Theorem 3.1. *For any measurable space (X, Σ) the space $M(X, \mathbf{2})$ is Baire.*

Proof. (1) We claim that X has the M -property. Let $\{\Delta_n : n \in \mathbb{N}\}$ be a pairwise disjoint sequence of finite subsets $\Delta_n \subset X$, $\Delta_n = A_n \cup B_n$, $A_n \cap B_n = \emptyset$, $n \in \mathbb{N}$.

Let $x, y \in \bigcup_{n=1}^{\infty} \Delta_n$ and $x \neq y$. Since the sets $\{x\}$ and $\{y\}$ are Σ -separated, there is $C_y \in \Sigma$ such that $x \in C_y$ and $y \in X \setminus C_y$. Let $C(x) = \bigcap \{C_y : y \in \bigcup_{n=1}^{\infty} \Delta_n \setminus \{x\}\}$. Since Σ is a σ -algebra, $C(x) \in \Sigma$. Note that $C(x) \cap C(y) = \emptyset$ for $x \neq y$. Let $C = \bigcup \{C(x) : x \in \bigcup_{n=1}^{\infty} A_n\}$.

Then $C \in \Sigma$. Note that $\bigcup_{n=1}^{\infty} A_n \subseteq C$ and $\bigcup_{n=1}^{\infty} B_n \subseteq X \setminus C$.

(2). We claim that the space $M(X, \mathbf{2})$ is Baire. Suppose the opposite. Let $M(X, \mathbf{2}) = \bigcup_{i=1}^{\infty} F_i$, where F_i is a closed nowhere dense subset of $M(X, \mathbf{2})$ and $F_i \subset F_{i+1}$ for every $i \in \mathbb{N}$.

Let U_1 be a non-empty base open set such that $U_1 \cap F_1 = \emptyset$. Denote by $\Theta_1 = \text{supp} U_1$. By Lemma 2, for a finite subset Θ_1 of X and nowhere dense set F_2 in $M(X, \mathbf{2})$ there is a base open set U_2 such that

- (1) $\text{supp} U_2 \cap \Theta_1 = \emptyset$;
- (2) for any $f \in U_2$ there is $\varphi \in \mathbf{2}^{\Theta_2}$, where $\Theta_2 := \text{supp} U_2 \cup \Theta_1$ such that $\varphi = f \upharpoonright \Theta_2$ and $\varphi \notin \overline{\{h \upharpoonright \Theta_2 : h \in F_2\}}^{2^{\Theta_2}}$.

For every $k \in \mathbb{N}$, by Lemma 2.2, for the finite set Θ_k in X and nowhere dense set F_{k+1} in $M(X, \mathbf{2})$ there is a base open set U_{k+1} such that

- (1) $\text{supp} U_{k+1} \cap \Theta_k = \emptyset$;
- (2) for any $f \in U_{k+1}$ there is $\varphi \in \mathbf{2}^{\Theta_{k+1}}$ where $\Theta_{k+1} = \text{supp} U_{k+1} \cup \Theta_k$ such that $\varphi = f \upharpoonright \Theta_{k+1}$ and $\varphi \notin \overline{\{h \upharpoonright \Theta_{k+1} : h \in F_{k+1}\}}^{2^{\Theta_{k+1}}}$.

Let $\Delta_1 = \text{supp} U_1 = \Theta_1$ and $\Delta_k = \text{supp} U_k = \Theta_k \setminus \Theta_{k-1}$ for every $k > 1$.

For every $k \in \mathbb{N}$ consider $f_k \in U_k$. Define the map $f : \bigcup_{k=1}^{\infty} \Delta_k \rightarrow \{0, 1\}$ such that $f \upharpoonright \Delta_k = f_k \upharpoonright \Delta_k$ for every $k \in \mathbb{N}$.

Since X has the M -property, there are a subsequence $\{n_k : k \in \mathbb{N}\}$ and $\tilde{f} \in M(X, \mathbf{2})$ such that $\tilde{f} \upharpoonright \bigcup_{k=1}^{\infty} \Delta_{n_k} = f \upharpoonright \bigcup_{k=1}^{\infty} \Delta_{n_k}$ by Lemma 2.1. Then $\tilde{f} \notin F_{n_k}$ for every $k \in \mathbb{N}$. Since $M(X, \mathbf{2}) = \bigcup_{k=1}^{\infty} F_{n_k}$, we receive a contradiction. □

Corollary 3.1. Let $\{(X_\alpha, \Sigma_\alpha) : \alpha \in A\}$ be a family measurable spaces. Then $\prod_{\alpha \in A} M(X_\alpha, \mathbf{2})$ is Baire.

Proof. Let $\Sigma_\oplus = \{A \subset \bigoplus_{\alpha \in A} X_\alpha : A \cap X_\alpha \in \Sigma_\alpha \text{ for each } \alpha \in A\}$. Then $(\bigoplus_{\alpha \in A} X_\alpha, \Sigma_\oplus)$ is a measurable space.

We claim that $\prod_{\alpha \in A} M(X_\alpha, \mathbf{2}) \cong M(\bigoplus_{\alpha \in A} X_\alpha, \mathbf{2})$. To this end, we define a mapping $f \rightarrow f^*$ where $f \in \prod_{\alpha \in A} M(X_\alpha, \mathbf{2})$ and $f^* \in M(\bigoplus_{\alpha \in A} X_\alpha, \mathbf{2})$.

Let $f \in \prod_{\alpha \in A} M(X_\alpha, \mathbf{2})$. Then $f = (f_\alpha)$ where $f_\alpha \in M(X_\alpha, \mathbf{2})$ for each $\alpha \in A$.

Define $f^* : \bigoplus_{\alpha \in A} X_\alpha \rightarrow \{0, 1\}$ by letting $f^* \upharpoonright X_\alpha = f_\alpha$ for each $\alpha \in A$.

We claim that $f^* \in M(\bigoplus_{\alpha \in A} X_\alpha, \mathbf{2})$. Let $i \in \{0, 1\}$. Then $(f^*)^{-1}(i) \cap X_\alpha = (f_\alpha)^{-1}(i) \cap X_\alpha \in \Sigma_\alpha$ for each $\alpha \in A$. It follows that $(f^*)^{-1}(i) \in \Sigma_\oplus$ and, hence, $f^* \in M(\bigoplus_{\alpha \in A} X_\alpha, \mathbf{2})$.

The mapping $f \rightarrow f^*$ is a one-to-one mapping from $\prod_{\alpha \in A} M(X_\alpha, \mathbf{2})$ onto $M(\bigoplus_{\alpha \in A} X_\alpha, \mathbf{2})$. Furthermore, both the mapping and its inverse are continuous. Thus, the mapping is a homeomorphism.

By Theorem 2.1, $M(\bigoplus_{\alpha \in A} X_\alpha, \mathbf{2})$ is Baire, hence, the space $\prod_{\alpha \in A} M(X_\alpha, \mathbf{2})$ is Baire. $\square$

Recall that a mapping $\varphi : K \rightarrow M$ is called *almost open* if $\text{Int}\varphi(V) \neq \emptyset$ whenever non-empty open subset V of K . Note that irreducible mappings are almost open mappings that are defined on compact spaces. Also note that if $\varphi_\alpha : K_\alpha \rightarrow M_\alpha$ ($\alpha \in A$) are surjective almost open mappings then the product mapping $\prod_{\alpha \in A} \varphi_\alpha : \prod_{\alpha \in A} K_\alpha \rightarrow \prod_{\alpha \in A} M_\alpha$ is also almost open.

Lemma 3.1. (Lemma 3.14 in [8]) Let $\psi : P \rightarrow L$ be a surjective continuous almost open mapping and $E \subset P$ be a dense non-meager (Baire) subspace in P . Then $\psi(E)$ is non-meager (Baire).

Lemma 3.2. ([1]) Let L be a metrizable compact space without isolated points, T be the Cantor set. Then there exists a surjective continuous irreducible mapping $\psi : T \rightarrow L$.

Lemma 3.3. Let $\psi : K \rightarrow L$ be a continuous almost open surjective mapping. If $M(X, K)$ is a dense subset of K^X and $M(X, K)$ is Baire, then $M(X, L)$ is Baire, too.

Proof. The mapping $\psi^X : K^X \rightarrow L^X$ is continuous almost open surjective. Since $M(X, K)$ is a dense subset of K^X and $M(X, K)$ is Baire, $\psi^X(M(X, K))$ is Baire by Lemma 3.1. Note that $\psi^X(M(X, K)) = \{\psi \circ f : f \in M(X, K)\} \subseteq M(X, L)$. Thus, $M(X, L)$ contains a dense Baire subset $\psi^X(M(X, K))$. It follows that $M(X, L)$ is Baire, too. $\square$

Theorem 3.2. Let (X, Σ) be a measurable space. Then the space $M(X, L)$ is Baire for any metrizable compact space L .

Proof. Note that $M(X, \mathbf{2}^\omega) = (M(X, \mathbf{2}))^\omega$. Thus, by Corollary 3.1, $M(X, \mathbf{2}^\omega)$ is Baire.

Let L be a metrizable compact space and let L_1 be a metrizable compact space without isolated points. By Lemma 3.2, there is a surjective continuous irreducible mapping $\psi : \mathbf{2}^\omega \rightarrow L \times L_1$. Let $\pi_L : L \times L_1 \rightarrow L$ be the projection mapping on L .

Then $\pi_L \circ \psi : \mathbf{2}^\omega \rightarrow L$ is a surjective continuous almost open mapping. By Lemma 3.3, $M(X, L)$ is Baire. $\square$

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