

Inverse Systems of Intuitionistic Fuzzy Topological Spaces of Soft Sets

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Abstract

In this paper, inverse systems of intuitionistic fuzzy soft topological spaces are defined. The existence and uniqueness of the inverse limit of such system are proved. It is shown that the inverse limit is a functor and that injective surjective morphisms are preserved.

Keywords: soft set, intuitionistic fuzzy topological spaces, inverse systems of intuitionistic fuzzy topological spaces, inverse limit.

2010 MSC: 54A40, 08A05, 18A30.

1. Introduction

Soft set theory was first introduced by Molodtsov [14] in 1999 as a new mathematical tool for handling uncertainty and vagueness. This concept has found many applications in decision making, medical diagnosis, and engineering.

To deal with hesitation and indeterminacy more effectively, intuitionistic fuzzy sets were combined with soft sets. Intuitionistic fuzzy sets, originally proposed by Atanassov [1], assign both membership and non-membership degrees to elements. The resulting intuitionistic fuzzy soft sets provide a powerful framework for modeling uncertain objects.

Topological structures on intuitionistic fuzzy soft sets were systematically introduced by Bayramov and Gunduz [3]. They defined the basic concepts of intuitionistic fuzzy soft topological spaces, including open and closed sets and continuous mappings. Subsequent studies have explored various properties of these spaces such as compactness, connectedness, and separation axioms [4, 15].

Inverse systems and their limits are fundamental tools in mathematics. They have been studied in ordinary topology as well as in fuzzy and soft topological spaces [11]. In particular, some authors have investigated inverse and direct systems in the category of soft topological spaces and fuzzy soft topological spaces.

However, a systematic study of inverse systems in the category of intuitionistic fuzzy topological spaces of soft sets has not yet been carried out. In this paper, we introduce the concept of inverse systems in this category. We prove that every inverse system has a limit, and this limit is unique. We also show that the inverse limit is a functor and that injective surjective morphisms are preserved. Moreover, we prove that if each space in the inverse system is soft strong Hausdorff and soft strong compact, then the inverse limit space is also a soft strong compact space.

These results extend the existing theory of intuitionistic fuzzy soft topological spaces and provide new categorical tools for studying families of uncertain topological structures.

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2. Preliminaries

In this section, we will give some preliminary information for the present study.

Definition 2.1. [14] Let X be an initial universe, E be a set of all parameters and $P(X)$ denotes the power set of X . By A we will denote a subset of E , i.e. $A \subset E$. A pair (F, A) is called a soft set over X , where F is a mapping given by $F : A \rightarrow P(X)$.

In other words, soft set is a parameterized family of subsets of the set X . For $e \in A$, $F(e)$ may be considered as the set of e -elements of the soft set (F, A) i.e.,

$$(F, A) = \{(e, F(e)) : e \in A \subseteq E, F : A \rightarrow P(X)\}.$$

Let $SS(X_E)$ denotes family of soft sets over X .

Definition 2.2. [6] A mappings $(\tau, \tau^*) : SS(X, E) \rightarrow [0, 1]$ is called an intuitionistic fuzzy topology on X if the following conditions are hold:

- a) $\tau(F, E) + \tau^*(F, E) \leq 1; \forall (F, E) \in SS(X, E);$
- b) $\tau(\Phi) = \tau(\tilde{X}) = 1, \tau^*(\Phi) = \tau^*(\tilde{X}) = 0$
- c) $\tau((F, E) \tilde{\cap} (G, E)) \geq \tau(F, E) \wedge \tau(G, E), \tau^*((F, E) \tilde{\cap} (G, E)) \leq \tau^*(F, E) \vee \tau^*(G, E),$
 $\forall (F, E), (G, E) \in SS(X, E)$
- d) $\tau\left(\bigcup_{i \in \Delta} (F_i, E)\right) \geq \bigwedge_{i \in \Delta} \tau(F_i, E), \tau^*\left(\bigcup_{i \in \Delta} (F_i, E)\right) \leq \bigvee_{i \in \Delta} \tau^*(F_i, E)$ for the family $\{(F_i, E)\}_{i \in \Delta}$.

The quadruple (X, E, τ, τ^*) is called intuitionistic fuzzy topological spaces of soft sets. The intuitionistic fuzzy topological space (X, E, τ, τ^*) is denoted by *IFTSS*.

Definition 2.3. [4] Let $(X, E, \tau, \tau^*), (Y, E', \sigma, \sigma^*)$ be two intuitionistic fuzzy soft topological spaces and $(f, \varphi) : (X, E, \tau, \tau^*) \rightarrow (Y, E', \sigma, \sigma^*)$ be a mapping. Then (f, φ) is called a continuous mapping at the soft point $x_e \in (X, E)$ if for each arbitrary soft set $(f, \varphi)(x_e) = (f(x))_{\varphi(e)} \in (G, E') \in SS(Y, E')$, there exists $(F, E) \in SS(X, E)$ such that $x_e \in (X, E), \tau(F, E) \geq \sigma(G, E'), \tau^*(F, E) \leq \sigma^*(G, E')$ and $(f, \varphi)|(F, E) \subset (G, E')$. If (f, φ) is a continuous mapping for each soft point, then (f, φ) is a continuous mapping.

Theorem 2.1. [4] $(f, \varphi) : (X, E, \tau, \tau^*) \rightarrow (Y, E', \sigma, \sigma^*)$ is a continuous mapping if and only if $\tau((f, \varphi)^{-1}(G, E')) \geq \sigma(G, E'), \tau^*((f, \varphi)^{-1}(G, E')) \leq \sigma^*(G, E')$ are satisfied, $\forall (G, E') \in SS(Y, E')$.

Definition 2.4. [6] Let (X, E, τ, τ^*) be an intuitionistic fuzzy topological space.

$(\beta, \beta^*) : SS(X, E) \rightarrow [0, 1]$ is called a base of (τ, τ^*) if the following conditions are holds: for $\forall (F, E) \in SS(X, E)$

$$\tau(F, E) = \bigvee_{\substack{(G_i, E) \in SS(X, E) \\ (G_i, E) \subseteq (F, E)}} \bigwedge_{i \in \Delta} \beta(G_i, E), \tau^*(F, E) = \bigwedge_{\substack{(G_i, E) \in SS(X, E) \\ (G_i, E) \subseteq (F, E)}} \bigvee_{i \in \Delta} \beta^*(G_i, E).$$

3. Inverse Systems of Intuitionistic Fuzzy Topological Spaces of Soft Sets

Let Λ be a directed set and *IFTSS* be all intuitionistic fuzzy topological spaces category.

Definition 3.1. Any functor $D : \Lambda^{op} \rightarrow IFTSS$ is called the inverse system of intuitionistic fuzzy topologies and the limit of D is called limit of the inverse system.

Theorem 3.1. Every inverse system has a limit in the category $IFTSS$ and this limit is unique.

Proof. Inverse system in $IFTSS$ category is defined:

$$\underline{X} = \left(\{X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*\}_{\alpha \in \Lambda}, \left\{ (p, q)_{\alpha}^{\alpha'} : (X_{\alpha'}, E_{\alpha'}, \tau_{\alpha'}, \tau_{\alpha'}^*) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*) \right\}_{\alpha < \alpha'} \right) \quad (3.1)$$

For each $r \in (0, 1]$

$$\underline{X}^{(r)} = \left(\{X_\alpha, E_\alpha, \tau_\alpha^r, \tau_\alpha^{*r}\}_{\alpha \in \Lambda}, \left\{ (p, q)_{\alpha}^{\alpha'} : (X_{\alpha'}, E_{\alpha'}, \tau_{\alpha'}^r, \tau_{\alpha'}^{*r}) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha^r, \tau_\alpha^{*r}) \right\}_{\alpha < \alpha'} \right) \quad (3.2)$$

is inverse system of soft bitopological spaces.

Thus we obtain two inverse systems of soft topological spaces as follows:

$$\left(\{X_\alpha, E_\alpha, \tau_\alpha^r\}, \left\{ (p, q)_{\alpha}^{\alpha'} : (X_{\alpha'}, E_{\alpha'}, \tau_{\alpha'}^r) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha^r) \right\}_{\alpha < \alpha'} \right) \quad (3.3)$$

$$\left(\{X_\alpha, E_\alpha, \tau_\alpha^{*r}\}, \left\{ (p, q)_{\alpha}^{\alpha'} : (X_{\alpha'}, E_{\alpha'}, \tau_{\alpha'}^{*r}) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha^{*r}) \right\}_{\alpha < \alpha'} \right) \quad (3.4)$$

The inverse systems (3.3) and (3.4) consists of the following two systems of sets

$$\left(\{X_\alpha\}_{\alpha \in \Lambda}, \left\{ p_{\alpha}^{\alpha'} : X_{\alpha'} \rightarrow X_\alpha \right\}_{\alpha < \alpha'} \right) \quad (3.5)$$

$$\left(\{E_\alpha\}_{\alpha \in \Lambda}, \left\{ q_{\alpha}^{\alpha'} : E_{\alpha'} \rightarrow E_\alpha \right\}_{\alpha < \alpha'} \right). \quad (3.6)$$

Inverse systems (3.3) and (3.4) have a limit in soft topological spaces category and this limit is denoted as $\left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau^r \right)$, $\left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau^{*r} \right)$. There the topology τ^r (τ^{*r})

is a restriction to a subspace $Y = \varprojlim X_\alpha$ of product topology $\prod_{\alpha \in \Lambda} \tau_\alpha^r \left(\prod_{\alpha \in \Lambda} \tau_\alpha^{*r} \right)$. Since for each $\alpha \in \Lambda$ $\tau_\alpha^r \subset \tau_\alpha^{*r}$, $\prod_{\alpha \in \Lambda} \tau_\alpha^r \subset \prod_{\alpha \in \Lambda} \tau_\alpha^{*r}$ is obtained. It follows that $\tau^r \subset \tau^{*r}$. Hence soft

bitopological space $\left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau^r, \tau^{*r} \right)$ is inverse limit of inverse system (3.2).

If $r > r' \in I_0$, then for each $\alpha \in \Lambda$, $\tau_\alpha^r \subset \tau_\alpha^{r'}$, $\tau_\alpha^{*r} \subset \tau_\alpha^{*r'}$. Then

$\prod_{\alpha} \tau_\alpha^r \subset \prod_{\alpha} \tau_\alpha^{r'}$, $\prod_{\alpha} \tau_\alpha^{*r} \subset \prod_{\alpha} \tau_\alpha^{*r'}$. Thus $\tau^r \subset \tau^{r'}$, $\tau^{*r} \subset \tau^{*r'}$.

Hence $\{(\tau^r, \tau^{*r})\}_{r \in I_0}$ is a descending family of soft bitopologies on $\varprojlim X_\alpha$. Then by using the family $\{(\tau^r, \tau^{*r})\}_{r \in I_0}$, gradation of openness $\tau : \varprojlim X_\alpha \rightarrow I$, $\tau^* : \varprojlim X_\alpha \rightarrow I$ are defined

$$\tau \left(F_{\varprojlim E_\alpha} \right) = \vee \left\{ r \in I_0 : F_{\varprojlim E_\alpha} \in \tau^r \right\}, \tau^* \left(F_{\varprojlim E_\alpha} \right) = \wedge \left\{ 1 - r : F_{\varprojlim E_\alpha} \in \tau^{*r} \right\}$$

It follows that $\tau \subset \tau^*$.

Let

$$\left\{ (P_\alpha, Q_\alpha) : \left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau_\alpha, \tau_\alpha^* \right) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*) \right\}_{\alpha \in \Lambda} \quad (3.7)$$

be a family of projections maps. Since

$$(P_\alpha, Q_\alpha) : \left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau_\alpha^r, \tau_\alpha^{*r} \right) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha^r, \tau_\alpha^{*r})$$

is continuous mapping of soft bitopological spaces for each $r \in I_0$, (3.7) is a family continuous of category *IFTSS*.

Now, let us show that the family $\left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau, \tau^*\right)$ is a unique limit of the inverse system (3.1).

It suffices to show that for every *IFTSS* (Z, E, σ, σ^*) and family of continuous mapping $\{(p_\alpha, q_\alpha) : (Z, E, \sigma, \sigma^*) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*)\}_{\alpha \in \Lambda}$ which satisfies the condition $(p_\alpha, q_\alpha) = (p_{\alpha'}^{\alpha'}, q_{\alpha'}^{\alpha'}) \circ (p_{\alpha'}, q_{\alpha'})$, $\forall \alpha < \alpha'$, there exists a unique continuous mapping

$$(\varphi, \psi) : (Z, E, \sigma, \sigma^*) \rightarrow \left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau, \tau^*\right)$$

which satisfies the condition $(P_\alpha, Q_\alpha) \circ (\varphi, \psi) = (p_\alpha, q_\alpha)$.

For each $r \in I_0$ $\{(p_\alpha, q_\alpha) : (Z, E, \sigma^r, \sigma^{*r}) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha^r, \tau_\alpha^{*r})\}_{\alpha \in \Lambda}$ is a family of continuous mappings of soft bitopological spaces. Since the inverse systems (3.3) and (3.4) have limits in the category of soft topological spaces, there exists a unique continuous mapping $(\varphi, \psi) : (Z, E, \sigma^r, \sigma^{*r}) \rightarrow \left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau^r, \tau^{*r}\right)$ which makes up the following commutative diagram:

$$\begin{array}{ccc} (Z, E) & \xrightarrow{(p_\alpha, q_\alpha)} & (X_\alpha, E_\alpha) \\ & \searrow (\varphi, \psi) & \nearrow (P_\alpha, Q_\alpha) \\ & (\varprojlim X_\alpha, \varprojlim E_\alpha) & \end{array}$$

The mapping (φ, ψ) is continuous mapping for each $r \in I_0$. Hence

$$(\varphi, \psi) : (Z, E, \sigma, \sigma^*) \rightarrow \left(\varprojlim X_\alpha, \varprojlim E_\alpha, \tau, \tau^*\right)$$

is a continuous mapping in category *IFTSS* and it is unique.

Now let us show that the operation of inverse limit is a functor in the category of *IFTSS*.

For this, we define limit of morphism of inverse systems. Let

$$(\underline{f}, \underline{g}) = \left(\varphi : \bar{\Lambda} \rightarrow \Lambda, \{(f_\alpha, g_\alpha) : X_{\varphi(\bar{\alpha})} \rightarrow Y_{\bar{\alpha}}\}_{\bar{\alpha} \in \bar{\Lambda}}\right)$$

be a morphism from the inverse system (3.1) to inverse system

$$\underline{Y} = \left(\{Y_{\bar{\alpha}}, \bar{E}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}^*\}_{\bar{\alpha} \in \bar{\Lambda}}, \{(p, q)_{\bar{\alpha}}^{\bar{\alpha}'} : (Y_{\bar{\alpha}'}, \bar{E}_{\bar{\alpha}'}, \bar{\tau}_{\bar{\alpha}'}, \bar{\tau}_{\bar{\alpha}'}^*) \rightarrow (Y_{\bar{\alpha}}, \bar{E}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}^*)\}_{\bar{\alpha} < \bar{\alpha}'}\right)$$

in the category of *IFTSS*.

For each $r \in I_0$

$$(\underline{f}^r, \underline{g}^r) = \left(\varphi : \bar{\Lambda} \rightarrow \Lambda, \{(f_{\bar{\alpha}}, g_{\bar{\alpha}}) : (X_{\varphi(\bar{\alpha})}, E_{\varphi(\bar{\alpha})}, \tau_{\varphi(\bar{\alpha})}^r, \tau_{\varphi(\bar{\alpha})}^{*r}) \rightarrow (Y_{\bar{\alpha}}, \bar{E}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}^r, \bar{\tau}_{\bar{\alpha}}^{*r})\}_{\bar{\alpha} \in \bar{\Lambda}}\right)$$

is a morphism of inverse system of soft bitopological spaces. This morphism induces continuous mappings of inverse limit spaces as

$$\varprojlim (\underline{f}^r, \underline{g}^r) : \left(\varprojlim (X_\alpha, E_\alpha, \tau_\alpha^r) \rightarrow \varprojlim (Y_{\bar{\alpha}}, \bar{E}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}^r)\right)$$

$$\varprojlim (\underline{f}^r, \underline{g}^r) : \left(\varprojlim (X_\alpha, E_\alpha, \tau_\alpha^{*r}) \rightarrow \varprojlim (Y_{\bar{\alpha}}, \bar{E}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}^{*r}) \right)$$

in the category of soft topological spaces. Hence for each $r \in I_0$ $\varprojlim (\underline{f}^r, \underline{g}^r)$ is continuous mapping of soft bitopological spaces. Thus

$$\varprojlim (\underline{f}, \underline{g}) : \left(\varprojlim (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*) \rightarrow \varprojlim (Y_{\bar{\alpha}}, \bar{E}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}, \bar{\tau}_{\bar{\alpha}}^*) \right)$$

is a continuous mapping of *IFTSS*.

Inverse systems and their morphisms construct a category in *IFTSS*. This category is denoted as *Inv(IFTSS)*.

Limit of inverse system (3.1) denoted as (X, E, τ, τ^*) .

For each $\alpha \in \Lambda$ $(\pi_\alpha, q_\alpha) : (X, E, \tau, \tau^*) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*)$ projection mapping are continuous and

$$\begin{array}{ccc} (X, E, \tau, \tau^*) & \xrightarrow{(\pi_\alpha, q_\alpha)} & (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*) \\ & \searrow (\pi_{\alpha'}, q_{\alpha'}) \quad \nearrow (p_{\alpha'}, q_{\alpha'}^*) & \\ & (X_{\alpha'}, E_{\alpha'}, \tau_{\alpha'}, \tau_{\alpha'}^*) & \end{array}$$

diagram is commutative.

Theorem 3.2. $\varprojlim : \text{Inv}(\text{IFTSS}) \rightarrow \text{IFTSS}$ is a functor.

Lemma 3.1. If (β, β^*) is a base of (X, E, τ, τ^*) *IFTSS*, then for $\forall r \in I_o$, (β^r, β^{*r}) is base of $(X, E, \tau^r, \tau^{*r})$ soft bitopology space.

Theorem 3.3. Let $(X, E, \tau, \tau^*) = \varprojlim_\alpha (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*)$, then family

$$(\beta, \beta^*) = \{(\pi_\alpha, q_\alpha)^{-1}(F_\alpha, E_\alpha), (\pi_\alpha, q_\alpha)^{-1}(G_\alpha, E_\alpha) \mid \alpha \in \Lambda, (F_\alpha, E_\alpha) \in \tau_\alpha, (G_\alpha, E_\alpha) \in \tau_\alpha^*\}$$

is base of soft topologies (τ, τ^*) .

Proof. Since $(\pi_\alpha, q_\alpha) : (X, E, \tau, \tau^*) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*)$ is a soft continuous mapping for all $\alpha \in \Lambda$, the sets of (β^r, β^{*r}) are soft open sets, for each $\forall r \in I_o$. It suffices to show that for every soft open $(H, E) \in (X, E, \tau^r, \tau^{*r})$ and every soft point $x_e \in (H, E)$, there exist $\alpha_0 \in \Lambda$ and

$$(F_{\alpha_0}, E_{\alpha_0}) \in \tau_{\alpha_0}^r, (G_{\alpha_0}, E_{\alpha_0}) \in \tau_{\alpha_0}^{*r}$$

such that

$$x_e \in (\pi_{\alpha_0}, q_{\alpha_0})^{-1}(F_{\alpha_0}, E_{\alpha_0}) \subset (H, E), x_e \in (\pi_{\alpha_0}, q_{\alpha_0})^{-1}(G_{\alpha_0}, E_{\alpha_0}) \subset (H, E).$$

Let (H, E) be an arbitrary soft open set in $(X, E, \tau^r, \tau^{*r})$ and $x_e = \{x_{e_\alpha}^\alpha\} \in (H, E)$. By the definition of soft subspace, there exists a soft open set $(P, E) \in \prod (X_\alpha, E_\alpha, \tau_\alpha^r, \tau_\alpha^{*r})$ such that $(H, E) = (X, E, \tau^r, \tau^{*r}) \cap (P, E)$. Thus, we have $\alpha_1, \alpha_2, \dots, \alpha_k \in \Lambda$ and $(F_{\alpha_i}, E_{\alpha_i}) \in \tau_{\alpha_i}^r, (G_{\alpha_i}, E_{\alpha_i}) \in \tau_{\alpha_i}^{*r}$ such that

$$\begin{aligned} x_e &\in (p_{\alpha_1}, q_{\alpha_1})^{-1}(F_{\alpha_1}, E_{\alpha_1}) \cap \dots \cap (p_{\alpha_k}, q_{\alpha_k})^{-1}(F_{\alpha_k}, E_{\alpha_k}) \in (H, E), \\ x_e &\in (p_{\alpha_1}, q_{\alpha_1})^{-1}(G_{\alpha_1}, E_{\alpha_1}) \cap \dots \cap (p_{\alpha_k}, q_{\alpha_k})^{-1}(G_{\alpha_k}, E_{\alpha_k}) \in (H, E). \end{aligned}$$

Since Λ is a directed set, there exists a $\alpha_0 \in \Lambda$ such that $\alpha_0 \succ \alpha_1, \dots, \alpha_0 \succ \alpha_k$. Also, since $(p_{\alpha_i}^{\alpha_0}, q_{\alpha_i}^{\alpha_0}) : (X_{\alpha_0}, E_{\alpha_0}, \tau_{\alpha_0}^r, \tau_{\alpha_0}^{*r}) \rightarrow (X_{\alpha_i}, E_{\alpha_i}, \tau_{\alpha_i}^r, \tau_{\alpha_i}^{*r})$ are soft continuous mappings for all $1 \leq i \leq k$,

$$\bigcap_{i=1}^k (p_{\alpha_i}^{\alpha_0}, q_{\alpha_i}^{\alpha_0})^{-1} (F_{\alpha_i}, E_{\alpha_i}) = (F_{\alpha_0}, E_{\alpha_0}), \quad \bigcap_{i=1}^k (p_{\alpha_i}^{\alpha_0}, q_{\alpha_i}^{\alpha_0})^{-1} (G_{\alpha_i}, E_{\alpha_i}) = (G_{\alpha_0}, E_{\alpha_0})$$

are soft open sets in $(X_{\alpha_0}, E_{\alpha_0}, \tau_{\alpha_0}^r, \tau_{\alpha_0}^{*r})$.

Since

$$(p_{\alpha_i}^{\alpha_0}, q_{\alpha_i}^{\alpha_0}) (x_{e_{\alpha_0}}^{\alpha_0}) = x_{e_{\alpha_i}}^{\alpha_i}, x_{e_{\alpha_0}}^{\alpha_0} \in (F_{\alpha_0}, E_{\alpha_0}), \quad x_{e_{\alpha_0}}^{\alpha_0} \in (G_{\alpha_0}, E_{\alpha_0})$$

and

$$x_e \in (\pi_{\alpha_0}, q_{\alpha_0})^{-1} (F_{\alpha_0}, E_{\alpha_0}), \quad x_e \in (\pi_{\alpha_0}, q_{\alpha_0})^{-1} (G_{\alpha_0}, E_{\alpha_0}),$$

we have

$$\begin{aligned} (\pi_{\alpha_0}, q_{\alpha_0})^{-1} (p_{\alpha_i}^{\alpha_0}, q_{\alpha_i}^{\alpha_0})^{-1} (F_{\alpha_i}, E_{\alpha_i}) &= (\pi_{\alpha_i}, q_{\alpha_i})^{-1} (F_{\alpha_i}, E_{\alpha_i}) = \\ &= \prod (X_{\alpha}, E_{\alpha}, \tau_{\alpha}^r, \tau_{\alpha}^{*r}) \cap (p_{\alpha_i}, q_{\alpha_i})^{-1} (F_{\alpha_i}, E_{\alpha_i}), \\ (\pi_{\alpha_0}, q_{\alpha_0})^{-1} (p_{\alpha_i}^{\alpha_0}, q_{\alpha_i}^{\alpha_0})^{-1} (G_{\alpha_i}, E_{\alpha_i}) &= (\pi_{\alpha_i}, q_{\alpha_i})^{-1} (G_{\alpha_i}, E_{\alpha_i}) = \\ &= \prod (X_{\alpha}, E_{\alpha}, \tau_{\alpha}^r, \tau_{\alpha}^{*r}) \cap (p_{\alpha_i}, q_{\alpha_i})^{-1} (G_{\alpha_i}, E_{\alpha_i}). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} x_e \in (\pi_{\alpha_0}, q_{\alpha_0})^{-1} (F_{\alpha_0}, E_{\alpha_0}) &= (\pi_{\alpha_0}, q_{\alpha_0})^{-1} \left(\bigcap_{i=1}^k (p_{\alpha_i}^{\alpha_0}, q_{\alpha_i}^{\alpha_0})^{-1} (F_{\alpha_i}, E_{\alpha_i}) \right) = \\ &= \bigcap_{i=1}^k \left(\prod (X_{\alpha}, E_{\alpha}, \tau_{\alpha}^r, \tau_{\alpha}^{*r}) \cap (p_{\alpha_i}, q_{\alpha_i})^{-1} (F_{\alpha_i}, E_{\alpha_i}) \right) = \\ &= \prod (X_{\alpha}, E_{\alpha}, \tau_{\alpha}^r, \tau_{\alpha}^{*r}) \cap \left(\bigcap_{i=1}^k (p_{\alpha_i}, q_{\alpha_i})^{-1} (F_{\alpha_i}, E_{\alpha_i}) \right) \subset \\ &\prod (X_{\alpha}, E_{\alpha}, \tau_{\alpha}^r, \tau_{\alpha}^{*r}) \cap (P, E) = (H, E). \end{aligned}$$

Definition 3.2. a) $(f, \varphi) : (X, E, \tau, \tau^*) \rightarrow (Y, E', \sigma, \sigma^*)$ is open, if for $\forall (F, E) \in SS(X, E)$

$$\tau(F, E) \leq \sigma((f, \varphi)(F, E)), \quad \tau^*(F, E) \geq \sigma^*((f, \varphi)(F, E))$$

b) $(f, \varphi) : (X, E, \tau, \tau^*) \rightarrow (Y, E', \sigma, \sigma^*)$ is closed, if for $\forall (F, E) \in SS(X, E)$

$$\tau(F, E)^c \leq \sigma((f, \varphi)(F, E))^c, \quad \tau^*(F, E)^c \geq \sigma^*((f, \varphi)(F, E))^c$$

Lemma 3.2. Let $(f, \varphi) : (X, E, \tau, \tau^*) \rightarrow (Y, E', \sigma, \sigma^*)$ be a mapping in the IFTS category.

a) (f, φ) is an open mapping if and only if $(f, \varphi) : (X, E, \tau^r, \tau^{*r}) \rightarrow (Y, E', \sigma^r, \sigma^{*r})$ is an open mapping in soft topology spaces for each $r \in I_0$.

b) (f, φ) is a closed mapping if and only if $(f, \varphi) : (X, E, \tau^r, \tau^{*r}) \rightarrow (Y, E', \sigma^r, \sigma^{*r})$ is a closed mapping in soft topology spaces for each $r \in I_0$.

Proof. a) Suppose $(f, \varphi) : (X, E, \tau, \tau^*) \rightarrow (Y, E', \sigma, \sigma^*)$ is an open mapping. Then for each $(F, E) \in (X, E)$, $\tau(F, E) \leq \sigma(f(F, E))$ and $\tau^*(F, E) \geq \sigma^*(f(F, E))$. Let us show that $(f, \varphi) : (X, E, \tau^r) \rightarrow (Y, E', \sigma^r)$ and $(f, \varphi) : (X, E, \tau^{*r}) \rightarrow (Y, E', \sigma^{*r})$ are open for each $r \in I_0$. For each $(G, E) \in \tau^r$, $\tau(G, E) \geq r$. We have $\sigma(f(G, E)) \geq \tau(G, E) \geq r$, i.e., $f(G, E) \in \sigma^r$. For each $(G, E) \in \tau^{*r}$, $\tau^*(G, E) \leq 1 - r$. Then $\sigma^*(f(G, E)) \leq \tau^*(G, E) \leq 1 - r$, i.e., $f(G, E) \in \sigma^{*r}$.

Conversely, assume the condition holds. Let us take arbitrary set $(F, E) \in (X, E)$. If $\tau(F, E) = 0$, then $\sigma(f(F, E)) \geq 0$, i.e., $\sigma(f(F, E)) \geq \tau(F, E)$. Similarly, if $\tau^*(F, E) = 1$, then $\sigma^*(f(F, E)) \leq 1$, i.e., $\sigma^*(f(F, E)) \leq \tau^*(F, E)$. This implies that (f, φ) is an open mapping.

Let $\tau(F, E) = r_0 > 0$ and $\tau^*(F, E) = r_1 < 1$, then $(F, E) \in \tau^{r_0}$ and $(F, E) \in \tau^{*(1-r_1)}$. Since $(f, \varphi) : (X, E, \tau^{r_0}) \rightarrow (Y, E', \sigma^{r_0})$ and $(f, \varphi) : (X, E, \tau^{*(1-r_1)}) \rightarrow (Y, E', \sigma^{*(1-r_1)})$ are open mappings, $f(F, E) \in \sigma^{r_0}$ and $f(F, E) \in \sigma^{*(1-r_1)}$, hence $\sigma(f(F, E)) \geq r_0 = \tau(F, E)$, $\sigma^*(f(F, E)) \leq 1 - (1 - r_1) = r_1 = \tau^*(F, E)$ i.e., (f, φ) is an open mapping.

b) Let $(f, \varphi) : (X, E, \tau, \tau^*) \rightarrow (Y, E', \sigma, \sigma^*)$ be a closed mapping. Then for each $(F, E) \in (X, E)$, $\tau(\underline{1} - (F, E)) \leq \sigma(\underline{1} - f(F, E))$ and $\tau^*(\underline{1} - (F, E)) \geq \sigma^*(\underline{1} - f(F, E))$ are satisfied. Now let us show that $(f, \varphi) : (X, E, \tau^r, \tau^{*r}) \rightarrow (Y, E', \sigma^r, \sigma^{*r})$ is closed in the soft topological spaces for each $r \in I_0$. For arbitrary two closed sets $F \in (\tau^r)'$ and $F^* \in (\tau^{*r})'$, $\tau(\underline{1} - F) \geq r$ and $\tau^*(\underline{1} - F^*) \leq 1 - r$. To complete the proof, we only need to check if $\underline{1} - f(F) \in \sigma^r$ and $\underline{1} - f(F^*) \in \sigma^{*r}$. Since $\sigma(\underline{1} - f(F)) \geq \tau(\underline{1} - F) \geq r$, $\sigma^*(\underline{1} - f(F^*)) \leq \tau^*(\underline{1} - F^*) \leq 1 - r$, $f(F) \in (\sigma^r)'$ and $f(F^*) \in (\sigma^{*r})'$ i.e., (f, φ) is a closed mapping of soft topological spaces.

Conversely, let $(f, \varphi) : (X, E, \tau^r, \tau^{*r}) \rightarrow (Y, E', \sigma^r, \sigma^{*r})$ be closed mapping of soft topological spaces for each $r \in I_0$. Let $(F, E) \in (X, E)$ be arbitrary set. If $\tau(\underline{1} - (F, E)) = r_0$ and $\tau^*(\underline{1} - (F, E)) = r_1$, then since $(f, \varphi) : (X, E, \tau^{r_0}) \rightarrow (Y, E', \sigma^{r_0})$ and $(f, \varphi) : (X, E, \tau^{*(1-r_1)}) \rightarrow (Y, E', \sigma^{*(1-r_1)})$ are closed mappings, $\underline{1} - f(F, E) \in \sigma^{r_0}$ and $\underline{1} - f(F, E) \in \sigma^{*(1-r_1)}$. Then $\sigma(\underline{1} - f(F, E)) \geq r_0 = \tau(\underline{1} - (F, E))$, $\sigma^*(\underline{1} - f(F, E)) \leq 1 - (1 - r_1) = r_1 = \tau^*(\underline{1} - (F, E))$ i.e., (f, φ) is a closed mapping.

Lemma 3.3. If $(f, \varphi) : (X, E, \tau, \tau^*) \rightarrow (Y, E', \sigma, \sigma^*)$ be an injective (surjective) mapping in *IFTS* category, then $(f, \varphi) : (X, E, \tau^r, \tau^{*r}) \rightarrow (Y, E', \sigma^r, \sigma^{*r})$ is an injective (surjective) mapping for each $r \in I_0$.

Theorem 3.4. Let

$$\underline{F} = \left\{ \wp : \Lambda' \rightarrow \Lambda, \{ (h_\beta, \nu_\beta) : X_{\wp(\beta)} \rightarrow Y_\beta \}_{\beta \in \Lambda'} \right\} \quad (3.8)$$

be a morphism of inverse systems in *IFTS* category. If (h_β, ν_β) is an injective (surjective) mapping for each $\forall \beta \in \Lambda'$, then $\varprojlim \bar{F} : \varprojlim \underline{X} \rightarrow \varprojlim \underline{Y}$ is also an injective (surjective) mapping.

Proof. For each $r \in I_0$,

$$\underline{F}^r = \left\{ \wp : \Lambda' \rightarrow \Lambda, \{ (h_\beta, \nu_\beta)^r : (X_{\wp(\beta)}, E, \tau_{\wp(\beta)}^r, \tau_{\wp(\beta)}^{*r}) \rightarrow (Y_\beta, E', \sigma_\beta^r, \sigma_\beta^{*r}) \}_{\beta \in \Lambda'} \right\}$$

is a morphism of inverse systems of intuitionistic soft bitopological spaces. Since $(h_\beta, \nu_\beta)^r$ is injective (surjective) mapping for each $\forall \beta \in \Lambda'$, $\varprojlim \bar{F}^r$ is injective (surjective) intuitionistic soft continuous mapping from [11, Theorem 2.1, Theorem 2.4]. Then $\varprojlim \bar{F} : \varprojlim \underline{X} \rightarrow \varprojlim \underline{Y}$ is injective (surjective) mapping.

Theorem 3.5. If inverse system in (3.1), (p_α', q_α') is an injective (surjective) mapping for all $\alpha \in \Lambda$, then $(\pi_\alpha, q_\alpha) : (X, E, \tau, \tau^*) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*)$ is an injective (surjective) mapping.

Proof. For the soft points $x_e = \{x_{e_\alpha}^\alpha\} \neq y_{e'} = \{y_{e'_\alpha}^\alpha\} \in (X, E, \tau, \tau^*)$, let $(\pi_{\alpha_1}, q_{\alpha_1})(x_e) = (\pi_{\alpha_1}, q_{\alpha_1})(y_{e'})$. For $\alpha' \succ \alpha_1$, since $(p_{\alpha_1'}^\alpha, q_{\alpha_1'}^\alpha) : (X_{\alpha'}, E_{\alpha'}, \tau_{\alpha'}, \tau_{\alpha'}^*) \rightarrow (X_{\alpha_1}, E_{\alpha_1}, \tau_{\alpha_1}, \tau_{\alpha_1}^*)$ is an

injective mapping and $(p_{\alpha_1}^{\alpha'}, q_{\alpha_1}^{\alpha'}) (x_{e_{\alpha_1}}^{\alpha'}) = x_{e_{\alpha_1}}^{\alpha_1} = y_{e_{\alpha_1}}^{\alpha_1} = (p_{\alpha_1}^{\alpha'}, q_{\alpha_1}^{\alpha'}) (y_{e_{\alpha_1}}^{\alpha'})$, we have $x_{e_{\alpha_1}}^{\alpha'} = y_{e_{\alpha_1}}^{\alpha'}$. For an arbitrary $\alpha \in \Lambda$, since Λ is a directed set, there exists $\alpha'' \in \Lambda$ such that $\alpha'' \succ \alpha$, $\alpha'' \succ \alpha_1$ for all $\alpha, \alpha_1 \in \Lambda$. Then $x_{e_{\alpha_1}}^{\alpha_1} = y_{e_{\alpha_1}}^{\alpha_1}$ implies that $x_{e_{\alpha''}}^{\alpha''} = y_{e_{\alpha''}}^{\alpha''}$. Hence $x_{e_{\alpha}}^{\alpha} = (p_{\alpha}^{\alpha''}, q_{\alpha}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = (p_{\alpha}^{\alpha''}, q_{\alpha}^{\alpha''}) (y_{e_{\alpha''}}^{\alpha''}) = y_{e_{\alpha}}^{\alpha}$ is satisfied, i.e. $x_e = y_e$.

Now we want to show that the soft mapping $(\pi_{\alpha_1}, q_{\alpha_1}) : (X, E, \tau, \tau^*) \rightarrow (X_{\alpha_1}, E_{\alpha_1}, \tau_{\alpha_1}, \tau_{\alpha_1}^*)$ is a surjective soft mapping. Let $x_{e_{\alpha_1}}^{\alpha_1} \in (X_{\alpha_1}, E_{\alpha_1}, \tau_{\alpha_1}, \tau_{\alpha_1}^*)$ be an arbitrary soft point. For $\alpha' \succ \alpha_1$, we take $x_{e_{\alpha'}}^{\alpha'} = (p_{\alpha_1}^{\alpha'}, q_{\alpha_1}^{\alpha'})^{-1} (x_{e_{\alpha_1}}^{\alpha_1})$. Then there exists $\alpha' \in \Lambda$ such that $\alpha' \succ \alpha$ and $\alpha' \succ \alpha_1$ for all $\alpha \in \Lambda$. Hence we obtain soft point $x_{e_{\alpha}}^{\alpha} = (p_{\alpha}^{\alpha'}, q_{\alpha}^{\alpha'}) (x_{e_{\alpha'}}^{\alpha'}) = (p_{\alpha}^{\alpha'}, q_{\alpha}^{\alpha'}) (p_{\alpha_1}^{\alpha'}, q_{\alpha_1}^{\alpha'})^{-1} (x_{e_{\alpha_1}}^{\alpha_1})$. Now we want to show that $x_e = \{x_{e_{\alpha}}^{\alpha}\} \in (X, E, \tau, \tau^*)$. We choose $\alpha', \tilde{\alpha}'$ such that $\alpha' \succ \alpha$, $\alpha' \succ \alpha_1$ and $\tilde{\alpha}' \succ \alpha$, $\tilde{\alpha}' \succ \alpha_1$ for all $\tilde{\alpha} \succ \alpha$. Thus,

$$\begin{aligned} x_{e_{\alpha}}^{\alpha} &= (p_{\alpha}^{\alpha'}, q_{\alpha}^{\alpha'}) (x_{e_{\alpha'}}^{\alpha'}) = (p_{\alpha}^{\alpha'}, q_{\alpha}^{\alpha'}) \left((p_{\alpha_1}^{\alpha'}, q_{\alpha_1}^{\alpha'})^{-1} (x_{e_{\alpha_1}}^{\alpha_1}) \right), \\ x_{e_{\tilde{\alpha}}}^{\tilde{\alpha}} &= (p_{\tilde{\alpha}}^{\tilde{\alpha}'}, q_{\tilde{\alpha}}^{\tilde{\alpha}'}) (x_{e_{\tilde{\alpha}'}}^{\tilde{\alpha}'}) = (p_{\tilde{\alpha}}^{\tilde{\alpha}'}, q_{\tilde{\alpha}}^{\tilde{\alpha}'}) \left((p_{\alpha_1}^{\tilde{\alpha}'}, q_{\alpha_1}^{\tilde{\alpha}'})^{-1} (x_{e_{\alpha_1}}^{\alpha_1}) \right). \end{aligned}$$

Now we take $\alpha'' \in \Lambda$ such that $\alpha'' \succ \alpha'$, $\alpha'' \succ \tilde{\alpha}'$. Then

$$x_{e_{\alpha_1}}^{\alpha_1} = (p_{\alpha_1}^{\alpha''}, q_{\alpha_1}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = (p_{\alpha_1}^{\alpha'}, q_{\alpha_1}^{\alpha'}) (p_{\alpha_1}^{\alpha''}, q_{\alpha_1}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = (p_{\alpha_1}^{\tilde{\alpha}'}, q_{\alpha_1}^{\tilde{\alpha}'}) (p_{\alpha_1}^{\alpha''}, q_{\alpha_1}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''})$$

and

$$x_{e_{\alpha_1}}^{\alpha_1} = (p_{\alpha_1}^{\alpha'}, q_{\alpha_1}^{\alpha'}) (x_{e_{\alpha'}}^{\alpha'}) = (p_{\alpha_1}^{\tilde{\alpha}'}, q_{\alpha_1}^{\tilde{\alpha}'}) (x_{e_{\tilde{\alpha}'}}^{\tilde{\alpha}'}).$$

Since the soft mappings $(p_{\alpha_1}^{\alpha'}, q_{\alpha_1}^{\alpha'})$, $(p_{\alpha_1}^{\tilde{\alpha}'}, q_{\alpha_1}^{\tilde{\alpha}'})$ are bijective mappings, $(p_{\alpha_1}^{\alpha''}, q_{\alpha_1}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = x_{e_{\alpha'}}^{\alpha'}$ and $(p_{\alpha_1}^{\alpha''}, q_{\alpha_1}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = (x_{e_{\tilde{\alpha}'}}^{\tilde{\alpha}'})$ are obtained. Thus,

$$(p_{\alpha}^{\alpha''}, q_{\alpha}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = (p_{\alpha}^{\alpha'}, q_{\alpha}^{\alpha'}) (p_{\alpha'}^{\alpha''}, q_{\alpha'}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = x_{e_{\alpha}}^{\alpha},$$

$$(p_{\alpha}^{\alpha''}, q_{\alpha}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = (p_{\alpha}^{\tilde{\alpha}'}, q_{\alpha}^{\tilde{\alpha}'}) (p_{\alpha'}^{\alpha''}, q_{\alpha'}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = x_{e_{\tilde{\alpha}}}^{\tilde{\alpha}}.$$

Hence $(p_{\alpha}^{\tilde{\alpha}'}, q_{\alpha}^{\tilde{\alpha}'}) (x_{e_{\tilde{\alpha}}}^{\tilde{\alpha}}) = (p_{\alpha}^{\alpha''}, q_{\alpha}^{\alpha''}) (x_{e_{\alpha''}}^{\alpha''}) = x_{\alpha}$ is obtained. Thus $x_e = \{x_{e_{\alpha}}^{\alpha}\} \in (X, E, \tau, \tau^*)$ and $(\pi_{\alpha_1}, q_{\alpha_1}) (x_e) = x_{e_{\alpha_1}}^{\alpha_1}$.

Corollary 3.1. If in (3.1), $(p_{\alpha}^{\alpha'}, q_{\alpha}^{\alpha'}) : (X_{\alpha'}, E_{\alpha'}, \tau_{\alpha'}, \tau_{\alpha'}^*) \rightarrow (X_{\alpha}, E_{\alpha}, \tau_{\alpha}, \tau_{\alpha}^*)$ is a homeomorphism for all $\alpha \in \Lambda$, then $(\pi_{\alpha}, q_{\alpha}) : (X, E, \tau, \tau^*) \rightarrow (X_{\alpha}, E_{\alpha}, \tau_{\alpha}, \tau_{\alpha}^*)$ is a homeomorphism.

Proof. By Theorem 3.5, $(\pi_{\alpha}, q_{\alpha})$ is a bijective and a soft continuous mapping. From theorem 3.3, the family

$$\{(\pi_{\alpha}, q_{\alpha})^{-1} (F_{\alpha}, E_{\alpha}), (\pi_{\alpha}, q_{\alpha})^{-1} (G_{\alpha}, E_{\alpha}) \mid \alpha \in \Lambda, (F_{\alpha}, E_{\alpha}) \in \tau, (G_{\alpha}, E_{\alpha}) \in \tau^*\}$$

is a base of soft topology of (X, E, τ, τ^*) . Then since $(\pi_{\alpha}, q_{\alpha}) ((\pi_{\alpha}, q_{\alpha})^{-1} (F_{\alpha}, E_{\alpha})) = (F_{\alpha}, E_{\alpha})$, $(\pi_{\alpha}, q_{\alpha}) ((\pi_{\alpha}, q_{\alpha})^{-1} (G_{\alpha}, E_{\alpha})) = (G_{\alpha}, E_{\alpha})$, $(\pi_{\alpha}, q_{\alpha})$ is a soft open mapping, $(\pi_{\alpha}, q_{\alpha})$ is a soft homeomorphism.

Definition 3.3. Let (X, E, τ, τ^*) be an *IFTS*. The space (X, E, τ, τ^*) is called strong Hausdorff space if and only if for soft points $x_e \neq y_{e'}$ and for each $r \in I_0$ there exist soft sets $(F, E), (G, E) \in SS(X, E)$ such that $x_e \in (F, E), y_{e'} \in (G, E), (F, E) \cap (G, E) = \emptyset$ and $\tau(F, E), \tau(G, E) \geq r; \tau^*(F, E), \tau^*(G, E) \leq 1 - r$.

It is clear that if (X, E, τ, τ^*) is strong Hausdorff space, then for each $r \in I_0$ $(X, E, \tau^r, \tau^{*r})$ soft bitopological space is Hausdorff space.

Definition 3.4. a) Let (X, E, τ, τ^*) be *IFTS* and $(G, E) \in (X, E)$ be a soft set. (G, E) is said to be smooth compact space if for each family of soft sets

$$(U, E) \in \{(F, E) : \tau(F, E) > 0, \text{ or } \tau^*(F, E) < 1\}$$

which satisfies the condition

$$\bigwedge_{x_e \in (X, E)} \left((G', E)(x_e) \vee \bigvee_{(F, E) \in (U, E)} (F, E)(x_e) \right) > p \quad \forall p \in I_1,$$

there exists a finite subfamily (H, E) of (U, E) such that

$$\bigwedge_{x_e \in (X, E)} \left((G', E)(x_e) \vee \bigvee_{(F, E) \in (H, E)} (F, E)(x_e) \right) > p.$$

If $(G, E) = \underline{1}$, then the space (X, E, τ, τ^*) is called smooth compact space.

b) If (G, E) is soft compact space in $(X, E, \tau^r, \tau^{*r})$ for each $r \in I_0$, then $(G, E) \in (X, E)$ is said to be strong smooth compact space.

Lemma 3.4. If the soft set (G, E) is smooth compact in *IFTS* (X, E, τ, τ^*) , then the soft set (G, E) is soft compact space in the soft topological space (X, E, τ^r) and (X, E, τ^{*r}) for each $r \in I_0$.

Proof. Let the soft set (G, E) be smooth compact in (X, E, τ, τ^*) . For each $r \in I_0$, let us consider $(U, E) = \{(F, E)\}$ which are strong p -shading soft sets of (G, E) in (X, E, τ^{*r}) for $\forall p \in I_1$, i.e., the following condition $\bigwedge_{x_e \in (X, E)} \left((G', E)(x_e) \vee \bigvee_{(F, E) \in (U, E)} (F, E)(x_e) \right) > p$ is satisfied. Since $(U, E) = \{(F, E)\}$ is a family of soft open sets, $\tau^*(F, E) \leq 1 - r$ for each $(F, E) \in (U, E)$. Since $r > 0$, $\tau^*(F, E) < 1$ or $\tau(F, E) > 0$ are satisfied. Then by using definition of smooth compact, it can be found a finite subfamily (H, E) of (U, E) such that $\bigwedge_{x_e \in (X, E)} \left((G', E)(x_e) \vee \bigvee_{(F, E) \in (H, E)} (F, E)(x_e) \right) > p$, i.e., (H, E) is strong p -shading of soft sets (G, E) . Hence from [16], the soft set (G, E) is soft compact in the soft space (X, E, τ^{*r}) . Since $\tau^r \subset \tau^{*r}$, the soft set (G, E) is soft compact in the soft space (X, E, τ^r) , too.

Theorem 3.6. If the inverse system

$$\underline{X} = \left(\{X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*\}_{\alpha \in \Lambda}, \left\{ (p, q)_\alpha^{\alpha'} : (X_{\alpha'}, E_{\alpha'}, \tau_{\alpha'}, \tau_{\alpha'}^*) \rightarrow (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*) \right\}_{\alpha < \alpha'} \right)$$

is smooth compact, strong Hausdorff in the category of *IFTS* and soft topologies $(\tau_\alpha^r, \tau_\alpha^{*r})$ are given by $\sup p$, for each $\alpha \in \Lambda, r \in I_0$, then $\varprojlim (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*)$ is strong smooth compact space.

Proof. From Definition 3.3, Definition 3.4 and Lemma 3.4, $(X_\alpha^r, E_\alpha^r, \tau_\alpha^r, \tau_\alpha^{*r})$ is inverse system of soft compact Hausdorff space for each $r \in I_0$. From [7] Theorem 3.17, $\varprojlim (X_\alpha^r, E_\alpha^r, \tau_\alpha^r, \tau_\alpha^{*r})$ is a soft compact space. Since $\varprojlim (X_\alpha^r, E_\alpha^r, \tau_\alpha^r, \tau_\alpha^{*r})$ is soft compact space for each $r \in I_0$, $\varprojlim (X_\alpha, E_\alpha, \tau_\alpha, \tau_\alpha^*)$ is strong smooth compact space in the category of *IFTS*.

4. Conclusion

In this paper, inverse systems in intuitionistic fuzzy soft topological spaces are introduced and studied. It is proved that every inverse system has a unique limit, and the inverse limit is a functor preserving injective and surjective morphisms. Moreover, the properties of soft strong Hausdorffness and soft strong compactness are preserved under inverse limits.

These results extend the existing theory and provide new tools for studying uncertain topological structures.

5. Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments and constructive suggestions that helped to improve the quality of this paper. The authors are also grateful to all readers who take the time to read and engage with this work.

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