

A New Approach to Universal Measurability

E.A. Reznichenko¹, Yu.V. Sadovnichiy^{*2}

^{1,2}*Department of Mathematics, Lomonosov Moscow State University, Moscow, Russia*

Abstract

V. Fedorchuk, A. Chizogidze, and T. Banakh in 2003 and V. Bogachev in 2024 posed the following questions: (i) is it true that $P_\tau(X)$ is C -embedded in $P_\sigma(X)$; (ii) Is it true that $P_R(X)$ is C -embedded in $P_R(\beta X)$ if and only if X is pseudocompact, where P_σ , P_τ , and P_R are the functors of probability σ -additive on the Baire σ -algebra, τ -additive, and Radon measures on the space X ? The answers to these questions are negative. However, if instead of probability measures we consider the corresponding alternating measures M_σ , M_τ , and M_R , the situation changes. It is proved that (i) $M_\tau(X)$ is C -embedded in $M_\sigma(X)$; (ii) $M_R(X)$ is C -embedded in $M_R(\beta X)$ if and only if X is pseudocompact.

The question of C -embedding of measure spaces is an extension of the question of coincidence of measure spaces, which is a development of the classical concepts of universally measurable and universal measure zero sets. A general theorem is obtained, which leads to the mentioned results.

Keywords: measure spaces, probability measures, Radon measures, universally measurable sets, universal measure zero sets.

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1. Introduction

This paper considers various ways of extending the concept of universally measurable and universal measure zero sets from separable metrizable spaces to the class of Tychonoff spaces.

Let $X \subset \mathbb{R}$.¹ A set X is called *universally measurable* if X is μ -measurable for any finite measure μ on the Borel sets of the line. A set X is called *universal measure-zero* if X has measure zero with respect to any continuous probability measure on Borel sets.

The classical extension of these concepts from the class of separable metrizable spaces to the class of Hausdorff spaces is Radon spaces and universally negligible spaces. These classes of spaces are inconvenient for applications, since not every compactum is a Radon space.

Given Ritz's theorem, for applications in analysis, functional analysis, and topology, it is more convenient to consider measures on the Baire σ -algebra of Tychonoff spaces. In [13], the concept of universally measurable spaces was extended from separable metrizable spaces to the class of Tychonoff spaces. In this case, all Hausdorff compact spaces turned out to be universally measurable.

Further, we assume that all spaces are Tychonoff.

1.1. Measures on a Tychonoff Space

^{*}Corresponding author: sadovnichiy.yu@gmail.com

¹In a slightly more general setting, we consider subsets of Polish spaces (=complete metric separable spaces).

For a space X , we denote by $\mathcal{B}(X)$ and $\mathcal{B}a(X)$ the σ -algebras of Borel and Baire sets of X .

For a compact space K , let $M(K)$ denote the dual of $C(K)$ with the weak* topology, where $C(K)$ is the Banach space of continuous functions on K with the sup-norm. By the fundamental theorem of Ritz, the functionals μ in $M(X)$ are identified with Borel regular measures on K . Denote $U(K)$ by the unit ball in $M(K)$, $M^+(K)$ by nonnegative functionals in $M(K)$, and $P(K) = \{\mu \in M^+(K) : \mu(K) = 1\}$ by probability measures on K .

For a Tychonoff space X , the most important measure spaces on X are: $M_a(X)$ by finitely additive measures on $\mathcal{B}a(X)$; $M_\sigma(X)$ — countably additive measures on $\mathcal{B}a(X)$; $M_\tau(X)$ — τ -additive measures on $\mathcal{B}(X)$; $M_R(X) = M(X)$ — Radon measures on $\mathcal{B}(X)$; $M_\beta(X)$ — compactly supported measures on $\mathcal{B}(X)$; $M_d(X)$ — discrete measures on $\mathcal{B}(X)$. These measure spaces are identified with subsets of $M(\beta X)$: for $k \in \{d, \beta, R, \tau, \sigma, a\}$ and $\mu \in M(\beta X)$, the inclusion $\mu \in M_k(X)$ holds if the condition (M_k) is satisfied:

- (M_d) μ is concentrated on some countable subset X ;
- (M_β) μ is concentrated on some compact subset X ;
- (M_R) μ is concentrated on some σ -compact subset X ;
- (M_τ) $|\mu|(K) = 0$ for all compact $K \subset \beta X \setminus X$;
- (M_σ) $|\mu|(K) = 0$ for all compact $K \subset \beta X \setminus X$ of type G_δ in βX ;
- (M_a) μ is arbitrary.

Note that

$$M_d(X) \subset M_R(X), \quad M_\beta(X) \subset M_R(X) \subset M_\tau(X) \subset M_\sigma(X) \subset M_a(X) = M(\beta X).$$

1.2. Generalization of Universally Measurable and Universal Measure Zero Sets

For a set $A \subset \mathbb{R}$, the classical notions of *universally measurable* and *universal measure zero sets* A are topological and are characterized as $M_\sigma(A) = M_R(A)$ and $M_\sigma(A) = M_d(A)$.

For $p, q \in \{a, \sigma, \tau, R, \beta, d\}$, the space X is called

- *universally $[p, q]$ -measurable* if $M_p(X) = M_q(X)$;
- *universally p -measurable* if $M_p(X) = M_R(X)$;
- *universally p -measure zero* if $M_p(X) = M_d(X)$.

Section 2 of the paper [13] is devoted to universally $[\sigma, \tau]$ -measurable and universally σ -measurable spaces (which are called universally measurable in [13]).

Let us formulate known results on this topic.

- (i) K -Analytic (in particular, Souslin) spaces are universally σ -measurable [14, 7.14.125].
- (ii) A space X is universally τ -measurable if and only if X is μ -measurable in βX for any measure $\mu \in P(\beta X)$ [14, Example 7.14.22].

(iii) A paracompact space X is universally $[\sigma, \tau]$ -measurable if and only if X does not contain an Ulam-measurable discrete closed subset (see [24, Theorem I.28] and [13, Section 2]). In particular, a Lindelöf space is a universally $[\sigma, \tau]$ -measurable space [13, Corollary 2.2].

(iv) For a space X , the following conditions are equivalent [4, Theorem 1.22]: (a) X is pseudocompact;

(b) X is a universally $[\sigma, a]$ -measurable space.

The following assertion is a strengthening of Remark 1(ii).

Theorem 1.1. *For a space X , the following statements are equivalent:*

- (i) X is universally τ -measurable;
- (ii) X is μ -measurable in βX for any measure $\mu \in P(\beta X)$;
- (iii) there exists an embedding of space X into some space Y such that X is μ -measurable in Y for any measure $\mu \in P(\beta Y)$;
- (iii) for any embedding of space X into any space Y , X is μ -measurable in Y for any measure $\mu \in P(\beta Y)$.

The Rudin–Knowles theorem [20, 18] states that for a compact space X , any Radon measure on X is discrete (i.e., $M_R(X) = M_d(X)$, which is equivalent to X being of universally R -measure zero) if and only if X is a sparse compact space. The Rudin–Knowles theorem implies the following assertion.

Proposition 1.1. A space X is of universally R -measure zero if and only if every compact subset of X is sparse.

Proposition 1.2. Let $p \in \{a, \sigma, \tau\}$. A space X is universally of p -measure zero if and only if X is universally p -measurable and every compact subset of X is sparse.

1.3. Functional Generalizations of Universally Measurable and Universally Measure Zero Sets

Recall that a subset Y of X is C -embedded (C^* -embedded) in X if every (bounded) continuous function on Y extends to a continuous function on X .

We consider the subfunctors $M_p^+(X) = M_p(X) \cap M^+(\beta X)$, $U_p(X) = M_p(X) \cap U(\beta X)$, $U_p^+(X) = U_p(X) \cap M^+(X)$, $P_p(X) = M_p(X) \cap P(\beta X)$ of non-negative measures, measures in norm not exceeding one, probability measures, respectively, of the functor $M_p(X)$. For $p, q \in \{a, \sigma, \tau, R, \beta, d\}$ and $Q \in \{M, M^+, U, U^+, P\}$, X is called

- *universally $Q_{[p,q]}$ -measurable ($Q_{[p,q]}^*$ -measurable)* if $Q_p(X)$ is C -embedded (C^* -embedded) in $Q_q(X)$;
- *universally Q_p -measurable (Q_p^* -measurable)* if X is universally $Q_{[R,p]}$ -measurable ($Q_{[R,p]}^*$ -measurable);
- *universally Q_p -measure (Q_p^* -measure) zero* if X is universally $Q_{[d,p]}$ -measurable ($Q_{[d,p]}^*$ -measurable).

Using the introduced concepts, some problems in measure theory and topology (see [4, Question 1.21, Question 1.29], [13, Section 5, Problem 5.5], [3, 2]) are special cases of the following general question.

Problem 1.1. For $p, q \in \{f, \sigma, \tau, R, \beta, d\}$ and $Q \in \{M, M^+, U, U^+, P\}$, describe universally $[a, b]$ -measurable, universally $Q_{[a,b]}$ -measurable, and universally $Q_{[a,b]}^*$ -measurable spaces.

In [13, Section 5], the problem is posed: to describe universally $P_{[\sigma,\tau]}$ -measurable and universally P_σ -measurable spaces. And the question arises [13, Problem 5.5]: is it true that every space X is universally $P_{[\sigma,\tau]}$ -measurable? In [4], a negative answer is given to this question.

In [3, 2], universally P_a^* -measurable spaces are studied, and it is proved that if X is universally P_a^* -measurable space, then X and $P(X)$ are pseudocompact spaces. In [4, Theorem 1.9], a converse of Bogochev's theorem was obtained: a space X is universally P_a^* -measurable if and only if $P(X)$ is a pseudocompact space. The question was posed: is it true that if X is pseudocompact, then X is universally P_a^* -measurable? In [4, Theorem 1.9], a negative answer was given to this question (see Examples 1.13 and 1.15 of [4, Theorem 1.9]).

Let us formulate known results on this topic.

(i) If X is a universally σ -measurable space and the finite powers X^n of X have a countable Souslin number (in particular, X is a Souslin space), then X^κ is a universally P_σ -measurable space for any cardinal κ [13, Corollary 5.2].

(ii) Any $AE(0)$ -space is a universally P_σ -measurable space [13, Corollary 5.3].

(iii) For a space X , the following conditions are equivalent [4, Theorem 1.22]: (a) X is pseudocompact; (b) X is a universally $[\sigma, a]$ -measurable space; (c) X is a universally $P_{[\sigma,a]}^*$ -measurable space.

(iii) For a space X , the following conditions are equivalent [4, Theorem 1.27]: (a) X is pseudo ω -bounded; (b) X is a universally $P_{[\beta,a]}^*$ -measurable space.

(iv) For a space X , the following conditions are equivalent [4, Theorem 1.9]: (a) $P(X)$ is pseudocompact; (b) X is a universally P_a^* -measurable space.

(v) For a space X with pregauge ω_1 , the following conditions are equivalent [4, Theorem 1.28]: (a) $P_\tau(X)$ is pseudocompact; (b) X is a universally $P_{[\tau,a]}^*$ -measurable space.

1.4. Alternating Measures

Probability measure functors have the best categorical properties. Other measure functors are also important [19, 5, 8, 21, 6, 7, 22] and are more convenient for some applications.

A subset M of a space X is called G_δ -dense if M intersects every nonempty G_δ -set (=the intersection of countably many open sets) in X .

A continuous mapping of compact spaces $f : X \rightarrow Y$ extends to a continuous mapping

$$M(f) : M(X) \rightarrow M(Y), \quad M(f)(\mu)(h) = \mu(h \circ f) \text{ for } h \in C(Y).$$

A continuous mapping of spaces $f : X \rightarrow Y$ extends to a continuous mapping $\beta f : \beta X \rightarrow \beta Y$ of Stone–Cech extensions. Then

$$M(\beta f)(M_p(X)) \subset M_p(Y),$$

where $p \in \{a, \sigma, \tau, R, \beta, d\}$.

The following theorem is the main result of the paper.

Theorem 1.2. Let X be a space and $p, q \in \{a, \sigma, \tau, R, \beta, d\}$. The following conditions are equivalent:

- (i) X is a universally $M_{[p,q]}$ -measurable space;
- (ii) X is a universally $M_{[p,q]}^*$ -measurable space;
- (iii) $M_p(X)$ is G_δ -dense in $M_q(X)$;
- (iv) For a measure $\mu \in M_q(X)$ and a sequence of functions $(h_n)_n \subset C(\beta X)$, there exists $\nu \in M_p(X)$ such that $\mu(h_n) = \nu(h_n)$ for all n ;
- (v)

$$M(\beta f)(M_q(X)) \subset M_p(Y),$$

for any separable metric space Y and continuous surjective mapping $f : X \rightarrow Y$.

From this theorem we obtain the following corollaries.

If X is pseudocompact, then X is universally M_a^* -measurable.

Any space is universally $M_{[\tau,\sigma]}$ -measurable.

1.5. Probabilistic, Non-Negative and Unit-Bounded Measures

There exists a pseudocompact locally compact space X that is neither universally P_σ^* -measurable nor $P_{[\tau,\sigma]}$ -measurable [4, Example 1.15], which answers negatively the questions in [13, 3]. As Corollaries 1 and 1 show, the answers to these questions are affirmative if we consider alternating measures rather than probability measures.

The following assertion follows directly from the definitions.

Proposition 1.3. Let X be a space, $p, q \in \{a, \sigma, \tau, R, \sigma, d\}$, and $Q \in \{M, M^+, U, U^+, P\}$. If X is a universally $Q_{[p,q]}$ -measurable space, then X is a universally $Q_{[p,q]}^*$ -measurable space.

Theorem 1.3. Let X be a space, $p, q \in \{a, \sigma, \tau, R, \beta, d\}$, and $Q \in \{M^+, U, P\}$. If X is a universally $Q_{[p,q]}^*$ -measurable space, then X is a universally $M_{[p,q]}$ -measurable space.

Theorem 1.4. Let X be a space and $p, q \in \{a, \sigma, \tau, R, \sigma, d\}$.

- (i) A space X is a universally $P_{[p,q]}$ -measurable space if and only if X is a universally $M_{[p,q]}^+$ -measurable space.
- (ii) A space X is a universally $P_{[p,q]}^*$ -measurable space if and only if X is a universally $M_{[p,q]}^{+*}$ -measurable space.

Problem 1.2. Let $p, q \in \{f, \sigma, \tau, R, \beta, d\}$. How are the following classes of spaces related?

- universally $P_{[p,q]}$ -measurable spaces;
- universally $P_{[p,q]}^*$ -measurable spaces;
- universally $U_{[p,q]}$ -measurable spaces;
- universally $U_{[p,q]}^*$ -measurable spaces;
- universally $M_{[p,q]}$ -measurable spaces.

2. Preliminary Information, Definitions and Notation

Let $\omega = \{0\} \cup \mathbb{N}$ be the first countable ordinal, and ω_1 be the first uncountable ordinal. By space, we mean a Tychonoff space.

2.1. Definitions from Measure Theory

Let μ be a finite (i.e., $\mu(X) < \infty$) σ -additive measure on the Borel σ -algebra $\mathcal{B}(X)$ of X .

Let $\mu = \mu^+ - \mu^-$ be the Hahn–Jordan decomposition. The measure $|\mu| = \mu^+ + \mu^-$ is called the total variation of μ . A measure μ is called *Radon* if for every Borel set $B \subset X$ and for every $\varepsilon > 0$, there exists a compact $K \subset B$ such that $|\mu|(B \setminus K) < \varepsilon$. The *support* $\text{supp } \mu$ of a measure μ is the smallest closed set $F \subset X$ for which $|\mu|(X \setminus F) = 0$. A measure μ is called *discrete* if $|\mu|(C) = 1$ for some countable $C \subset X$. A measure μ is called *continuous* if $\mu(\{x\}) = 0$ for every $x \in X$. A measure μ is called τ -*additive* if

$$|\mu|(\bigcup \mathcal{U}) = \sup\{|\mu|(\bigcup \mathcal{V}) : \mathcal{V} \subset \mathcal{U}, |\mathcal{V}| < \omega\}$$

for any family $\mathcal{U}$ of open subsets of X .

For a non-negative measure μ , we denote

$$\mu^*(A) = \inf\{\mu(B) : A \subset B \in \mathcal{B}(X)\}, \quad \mu_*(A) = \sup\{\mu(B) : A \supset B \in \mathcal{B}(X)\}$$

external and *internal* measures of a set $A \subset X$.

A countably additive measure μ on the σ -algebra $\mathcal{B}(X)$ of Borel sets in X is called a *Borel* measure. We denote the set of probability Borel measures on X by $P_b(X)$. Let $\mu \in P_b(X)$. A probability measure μ is called a *Radon* measure if, for every Borel set $B \subset X$ and for every $\varepsilon > 0$, there exists a compact $K \subset B$ such that $\mu(B \setminus K) < \varepsilon$. We denote the set of probability τ -additive measures on X by $P_\tau(X)$. Note that $P(X) \subset P_\tau(X)$.

Let Y be a space and $X \subset Y$. The set $P_b(X)$ is naturally embedded in $P_b(Y)$, and a measure $\mu \in P_b(X)$ corresponds to a measure $\tilde{\mu} \in P_b(Y)$ such that $\tilde{\mu}(B) = \mu(B \cap X)$ for $B \in \mathcal{B}(Y)$. We assume that $P_b(X) \subset P_b(Y)$. If Y is compact, then $P(X) \subset P_\tau(X) \subset P(Y)$.

Proposition 2.1 ([1]). Let Y be a compact space and $X \subset Y$. Then $w(X) = w(P(X)) = w(P_\tau(X))$ and

$$P(X) = \{\mu \in P(Y) : \mu_*(X) = 1\}, \quad P_\tau(X) = \{\mu \in P(Y) : \mu^*(X) = 1\}.$$

The embedding of $P(X)$ and $P_\tau(X)$ into $P(Y)$ is a topological embedding.

Let X be a space. Then

$$M_k(X) = \{t\mu - q\nu : \mu, \nu \in P_k(X), 0 \leq t, q\},$$

$$M_k(X) = \{t\mu : \mu \in P_k(X), 0 \leq t\},$$

$$U_k(X) = \{t\mu + q\nu : \mu, \nu \in P_k(X), 0 \leq t, q \leq 1\}.$$

for $k \in \{d, \beta, R, \tau, \sigma, a\}$. The space $U_k(X)$ is the unit ball in $M_k(X)$. The spaces $U_k(X)$ and $M_k(X)$ for $k \in \{\beta, R, \tau\}$ were studied in the works [19, 5, 8, 21, 6, 7, 22].

2.2. Definitions from Topology

If X is a space and $M \subset X$, then we denote by $\overline{M}^X$ the closure of M in X . If the context makes it clear which X is meant, we write $\overline{M}$ instead of $\overline{M}^X$.

A space X is said to have a countable Suslin number if every disjoint family of open sets in X is at most countable.

A space X is called ω -bounded if $\overline{M}$ is compact for every countable $M \subset X$. A space X is called *totally countably compact* if for every infinite $L \subset X$, there exists a countable $M \subset L$ such that $\overline{M}$ is compact (see [25]).

A space X is called *pseudo- ω -bounded* if for every sequence $(U_n)_n$ of open nonempty open sets, there exists a compact subset $K \subset X$ such that $U_n \cap K \neq \emptyset$ for all n (see [17, 10, 16]).

A space X is called *almost pseudo ω -bounded* (almost pseudo- ω -bounded) if, for any sequence $(U_n)_n$ of open nonempty open sets, there exists a subsequence $(n_k)_k$ and a compact subset $K \subset X$ such that $U_{n_k} \cap K \neq \emptyset$ for all k [10, Definition 3.8], [11, Definition 1.4.5].

A subset $G \subset X$ of X is called a G_δ -set if it is the intersection of countably many open sets.

A subset $S \subset X$ of X is called G_δ -dense in X if S intersects every nonempty G_δ -set. A space X is pseudocompact if and only if X is G_δ -dense in the Stone–Cech extension of βX .

A space X is called *realcompact* if X embeds closedly in the product of lines $\mathcal{R}^A$. A *Hewitt realcompactification* νX of X is a real complete extension of X such that X is dense in νX and X is C -embedded in νX (see [9, section 3.11]). The Hewitt extension νX is naturally embedded in the Stone–Cech extension βX : $p \in \nu X \subset \beta X$ if and only if every G_δ -set $G \subset \beta X$ containing p intersects X . The space X is pseudocompact if and only if $\nu X = \beta X$. The space X is real complete if and only if $\nu X = X$ (see [9, section 3.11]).

A subset $F \subset X$ is called a *null set* if $F = f^{-1}(0)$ for some continuous function on X .

A space X is called *perfectly κ -normal* if the closure of every open set is a null set.

3. Proofs

Let us describe the main steps of the proof.

Proposition 3.1. Let L_w be a locally convex linear space L with the weak topology. Then, for a dense subset $M \subset L_w$, the following conditions are equivalent:

1. M is C^* -embedded in L_w ;
2. M is C -embedded in L_w ;
3. M is G_δ -dense in L_w .

Proof. We show that L_w is perfectly κ -normal. Let $\widehat{L}_w$ be the completion of L_w . Then $\widehat{L}_w$ is linearly homeomorphic to a power of the line $\mathcal{R}^A$ [15, Corollary 3.12.43]. The space $\mathcal{R}^A$ is perfectly κ -normal [23]. Since L_w is dense in $\widehat{L}_w$, L_w is perfectly κ -normal.

(ii) follows from (i) and Lemma 9.9.35, Theorem 6.1.5, Proposition 6.1.6, Theorem 6.1.7, Lemma 8.4.5, and Theorem 9.9.36 from [12]. $\square$

Proof of Theorem 1.2. We assume that $p \neq q$.

Proposition 3.1 implies the equivalence of items (i), (ii), and (iii) of Theorem 1.2.

(iv) $\Leftrightarrow$ (v) is easily verified.

(iii) $\Leftrightarrow$ (v) We show that $M_q(Y)$ has a countable pseudocharacter. If $q = a$, then $M_p(X) \subset M_\sigma(X)$ and $M_\sigma(X)$ are G_δ -dense in $M(\beta X)$. Consequently, X is pseudocompact. Then Y is a metric compact set and $M_q(Y) = M(Y)$ has a countable network. If $q \neq a$, then $M_q(Y) \subset M_\tau(Y)$ has a countable network.

Since $M_p(X)$ is G_δ -dense in $M_q(X)$, it follows that $M(\beta f)(M_p(X))$ is G_δ -dense in $M(\beta f)(M_q(X)) \subset M_q(Y)$. Consequently, $M(\beta f)(M_p(X)) = M(\beta f)(M_q(X)) \subset M_q(Y)$.

(iv) $\Leftrightarrow$ (iii) Assume the contrary, $M_p(X)$ is not G_δ -dense in $M_q(X)$. Let $G \subset M_q(X) \setminus M_p(X)$ be a non-empty zero set and $\mu \in G$. Then there will be sequence of functions $(h_n)_n \subset C(\beta X)$ such that $F = \bigcap_n h_n^{-1}(h_n(\mu)) \subset G$. Since $F \subset M_q(X) \setminus M_p(X)$, then condition (iv) is violated. $\square$

Theorem 1.4 follows from the fact that $P_q(X)$ is C -embedded (C^* -embedded) in $P_p(X)$ if and only if $M_q^+(X)$ is C -embedded (C^* -embedded) in $M_p^+(X)$.

4. Conclusion

A concise summary of the main findings and their practical significance.

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