

About One Partial Mapping of Euclidean Space E_5

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Abstract

In the domain $\Omega \subset E_5$ consider a set of smooth lines such that through each point $X \in \Omega$ there passes exactly one line ω^1 from this set. The moving frame $\mathfrak{R}$ is a Frenet frame [1] associated with the line ω^1 of the given set. The integral lines of the vector fields $\vec{e}_i$ form the Frenet net Σ_5 [1]. There exists a point F_3^2 on the tangent of the line ω^3 of the Frenet net. When the point X moves within the domain Ω , the point F_3^2 describes out its domain Ω_3^2 in E_5 . The partial mapping is defined $f_3^2 : \Omega \rightarrow \Omega_3^2$ such that $f_3^2(X) = F_3^2$.

We consider the four-dimensional distributions $\Delta_4 = (X, \vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5)$ and $\Delta'_4 = f_3^2(\Delta_4)$. In the case where the Frenet is a cyclic Frenet net, the necessary and sufficient conditions for the existence of quasi-double lines of the pair (Δ_4, Δ'_4) of four-dimensional distributions under the partial mapping f_3^2 in the space E_5 are proven.

Keywords: Euclidean space, partial mapping, cyclic Frenet net, pseudofocus, a quasi-double line, distribution.

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1. Introduction

In the domain $\Omega \subset E_5$ consider a set of smooth lines such that through each point $X \in \Omega$ there passes exactly one line from the given set. The moving frame $\mathcal{R} = (X, \vec{e}_i)$ ($i, j, k = \overline{1, 5}$) is the Frenet frame for the line ω^1 of the given set of smooth lines. The derivation formula for the frame $\mathcal{R}$ has the following form:

$$d\vec{X} = \omega^i \vec{e}_i, \quad d\vec{e}_i = \omega_i^k \vec{e}_k. \quad (1.1)$$

The forms ω^i, ω_i^k satisfy the structure equations of Euclidean space:

$$D\omega^i = \omega^k \wedge \omega_k^i, \quad D\omega_i^k = \omega_i^j \wedge \omega_j^k, \quad \omega_i^j + \omega_j^i = 0. \quad (1.2)$$

Integral lines of the vector fields $\vec{e}_i$ form the net Σ_5 of Frenet [1] for the line ω^1 of the given set of lines. Since the frame $\mathcal{R}$ is constructed on tangents of the lines of the net Σ_5 , the forms ω_i^k are principal forms. In other words:

$$\omega_i^k = \Lambda_{ij}^k \omega^j. \quad (1.3)$$

Taking into account (1.3) from (1.2) we have:

$$\Lambda_{ij}^k = -\Lambda_{kj}^i. \quad (1.4)$$

If we differentiate equation (1.3) externally, then we have:

$$D\omega_i^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k D\omega^j.$$

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By using formula (1.2) we get:

$$\omega_i^j \wedge \omega_j^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k \omega^\ell \wedge \omega_\ell^j.$$

Taking into account (1.3) from the last formula it follows that:

$$\omega_i^j \wedge \Lambda_{j\ell}^k \omega^\ell = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \omega_\ell^j \wedge \omega^\ell.$$

or

$$\Lambda_{j\ell}^k \omega_i^j \wedge \omega^\ell = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \omega_\ell^j \wedge \omega^\ell.$$

From here, we find:

$$d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{i\ell}^k \omega_j^\ell \wedge \omega^j - \Lambda_{j\ell}^k \omega_i^j \wedge \omega^\ell = 0$$

or

$$(d\Lambda_{ij}^k - \Lambda_{i\ell}^k \omega_j^\ell - \Lambda_{\ell j}^k \omega_i^\ell) \wedge \omega^j = 0.$$

By using Cartan's lemma [2], we have:

$$d\Lambda_{ij}^k - \Lambda_{i\ell}^k \omega_j^\ell - \Lambda_{\ell j}^k \omega_i^\ell = \Lambda_{ijm}^k \omega^m$$

or

$$d\Lambda_{ij}^k = (\Lambda_{ijm}^k + \Lambda_{i\ell}^k \Lambda_{jm}^\ell + \Lambda_{\ell j}^k \Lambda_{im}^\ell) \omega^m. \quad (1.5)$$

The system of variable $\{\Lambda_{ij}^k, \Lambda_{ijm}^k\}$ forms a geometric object of the second order. The Frenet formulas for the line ω^1 of the given set have the following form:

$$d_1 \vec{e}_1 = \Lambda_{11}^2 \vec{e}_2,$$

$$d_1 \vec{e}_2 = \Lambda_{21}^1 \vec{e}_1 + \Lambda_{21}^3 \vec{e}_3,$$

$$d_1 \vec{e}_3 = \Lambda_{31}^2 \vec{e}_2 + \Lambda_{31}^4 \vec{e}_4,$$

$$d_1 \vec{e}_4 = \Lambda_{41}^3 \vec{e}_3 + \Lambda_{41}^5 \vec{e}_5,$$

$$d_1 \vec{e}_5 = \Lambda_{51}^4 \vec{e}_4,$$

and

$$\Lambda_{11}^3 = -\Lambda_{31}^1 = 0, \quad \Lambda_{11}^4 = -\Lambda_{41}^1 = 0, \quad \Lambda_{11}^5 = -\Lambda_{51}^1 = 0. \quad (1.6)$$

$$\Lambda_{21}^5 = -\Lambda_{51}^2 = 0, \quad \Lambda_{21}^4 = -\Lambda_{41}^2 = 0, \quad \Lambda_{31}^5 = -\Lambda_{51}^3 = 0. \quad (1.7)$$

There $k_1^1 = \Lambda_{11}^2$, $k_2^1 = \Lambda_{21}^3$, $k_3^1 = \Lambda_{31}^4$, $k_4^1 = \Lambda_{41}^5 = -\Lambda_{51}^4$ — are the first, second, third and fourth curvatures of the line ω^1 respectively (where d_1 denotes the derivative along the line ω^1).

A pseudofocus [3], [5] F_i^j (where $i \neq j$) of the tangent of the line ω^1 of Σ_5 is defined by the radius-vector:

$$\vec{F}_i^j = \vec{X} - \frac{1}{\Lambda_{ij}^j} \vec{e}_i = \vec{X} + \frac{1}{\Lambda_{ji}^i} \vec{e}_j. \quad (1.8)$$

There exist four pseudofoci on each tangent $(X, \vec{e}_i)$:

on the straight line $(X, \vec{e}_1) - F_1^2, F_1^3, F_1^4, F_1^5$;

on $(X, \vec{e}_2) - F_2^1, F_2^3, F_2^4, F_2^5$;

on $(X, \vec{e}_3) - F_3^1, F_3^2, F_3^4, F_3^5$;

on $(X, \vec{e}_4) - F_4^1, F_4^2, F_4^3, F_4^5$;

on $(X, \vec{e}_5) - F_5^1, F_5^2, F_5^3, F_5^4$.

The net $\tilde{\Sigma}_5$ in $\Omega \subset E_5$ is called a cycle net of Frenet [4] if the frames

$$\mathfrak{R}_1 = (X, \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5),$$

$$\mathfrak{R}_2 = (X, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5, \vec{e}_1),$$

$$\mathfrak{R}_3 = (X, \vec{e}_3, \vec{e}_4, \vec{e}_5, \vec{e}_1, \vec{e}_2),$$

$$\mathfrak{R}_4 = (X, \vec{e}_4, \vec{e}_5, \vec{e}_1, \vec{e}_2, \vec{e}_3),$$

$$\mathfrak{R}_5 = (X, \vec{e}_5, \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4).$$

are Frenet frames for the lines $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ of net Σ_5 , respectively, simultaneously.

Let the net Σ_5 be a Frenet cycle net [4], setting and solving the problem. We will denote it by $\tilde{\Sigma}_5$. The pseudofocus $F_3^2 \in (X, \vec{e}_3)$ is defined by the radius-vector:

$$\vec{F}_3^2 = \vec{X} - \frac{1}{\Lambda_{32}^2} \vec{e}_3. \quad (1.9)$$

When the point X is moving in the domain $\Omega \subset E_5$, the pseudofocus F_3^2 describes its domain $\Omega_3^2 \subset E_5$. Thus, we defined the partial mapping $f_3^2 : \Omega \rightarrow \Omega_3^2$ such that $f_3^2(X) = F_3^2$. We will associate the moving frame $\mathfrak{R}' = (F_3^2, \vec{c}_i) (i = \overline{1, 5})$, to the domain Ω_3^2 , where the vectors $\vec{c}_i$ have the form [4], [7]:

$$\begin{aligned} \vec{c}_1 &= \vec{e}_1 - \frac{\Lambda_{31}^2}{\Lambda_{32}^2} \vec{e}_2 + \frac{C_{321}^2}{(\Lambda_{32}^2)^2} \vec{e}_3 - \frac{\Lambda_{31}^4}{\Lambda_{32}^2} \vec{e}_4; \\ \vec{c}_2 &= \vec{e}_2 - \frac{C_{322}^2}{(\Lambda_{32}^2)^2} \vec{e}_3 - \frac{\Lambda_{32}^4}{\Lambda_{32}^2} \vec{e}_4; \\ \vec{c}_3 &= \left[1 + \frac{C_{323}^2}{(\Lambda_{32}^2)^2} \right] \vec{e}_3 - \frac{\Lambda_{33}^4}{\Lambda_{32}^2} \vec{e}_4; \\ \vec{c}_4 &= -\frac{\Lambda_{34}^2}{\Lambda_{32}^2} \vec{e}_2 + \frac{C_{324}^2}{(\Lambda_{32}^2)^2} \vec{e}_3 + \vec{e}_4; \\ \vec{c}_5 &= \frac{\Lambda_{35}^2}{\Lambda_{32}^2} \vec{e}_2 + \frac{C_{325}^2}{(\Lambda_{32}^2)^2} \vec{e}_3 + \vec{e}_5 - \frac{\Lambda_{35}^4}{\Lambda_{32}^2} \vec{e}_4. \end{aligned} \quad (1.10)$$

2. Setting and Solving the Problem

It is considered a pair of four dimensional distribution $\Delta_4 = (X, \vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5)$ and $\Delta'_4 = f_3^2(\Delta_4)$. Let the line θ belong to the distribution Δ_4 . Then the tangent vector of line θ has a form $\vec{\theta} = \theta^1 \vec{e}_1 + \theta^2 \vec{e}_2 + \theta^4 \vec{e}_4 + \theta^5 \vec{e}_5$. We find the tangent vector $\vec{\theta}$ of the line $\bar{\theta} = f_3^2(\theta) : \vec{\theta} = \theta^1 \vec{e}_1 + \theta^2 \vec{e}_2 + \theta^4 \vec{e}_4 + \theta^5 \vec{e}_5$. Taking into account formulas (1.10) from here we find:

$$\vec{\theta} = \theta^1 \vec{e}_1 + (\theta^1 c_1^2 + \theta^2 + \theta^4 c_4^2 + \theta^5 c_5^2) \vec{e}_2 + (\theta^1 c_1^3 + \theta^2 c_2^3 + \theta^4 c_4^3 + \theta^5 c_5^3) \vec{e}_3 + (\theta^1 c_1^4 + \theta^2 c_2^4 + \theta^4 + \theta^5 c_5^4) \vec{e}_4 + \theta^5 \vec{e}_5,$$

where c_i^j - is j -th coordinate of vectors $\vec{c}_i$.

Definition 2.1. If vectors $\vec{\theta}, \vec{\bar{\theta}}, \overrightarrow{XF_3^2}$ belong to the distribution $\Delta_4 = (X, \vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5)$, then the lines θ and $\bar{\theta}$ are called quasi-double lines of the partial mapping f_3^2 .

Since the vector $\overrightarrow{F_3^2} = -\frac{1}{\Lambda_{32}^2} \vec{e}_3$ does not belong to the distribution Δ_4 , then lines θ and $\bar{\theta}$ cannot be quasi-double lines of the partial mapping f_3^2 .

Definition 2.2. If the tangent of the line θ at the point X and the tangent of the line $\bar{\theta} = f_3^2(\theta)$ at the point F_3^2 belong to the same four dimensional space $(X, \vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5)$, then lines θ and $\bar{\theta}$ are called quasi-double lines of the pair $(\Delta_{(1245)}, \Delta'_{(1245)})$ in the partial mapping $f_3^2 : \Omega \rightarrow \Omega_3^2$.

3. Main Results

From the condition $\vec{\theta}, \vec{\bar{\theta}} \in (\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5)$ we have:

$$\theta^1 c_1^3 + \theta^2 c_2^3 + \theta^4 c_4^3 + \theta^5 c_5^3 = 0.$$

Taking into account formulas (1.10) from here we find:

$$\theta^1 C_{321}^2 + \theta^2 C_{322}^2 + \theta^4 C_{324}^2 + \theta^5 C_{325}^2 = 0. \quad (3.1)$$

If the coordinates of the tangent vector $\vec{\theta}$ of the θ satisfy the condition (3.1), the lines $\theta, \bar{\theta}$ are quasi-double of the pair $(\Delta_{(1245)}, \Delta'_{(1245)})$ of four dimensional distributions in the partial mapping f_3^2 .

Then the following theorem is proven:

Theorem 3.1. Lines $\theta \subset \Delta_{(1245)}$ and $\bar{\theta} = f_3^2(\theta)$ are quasi-double lines of the pair $(\Delta_{(1245)}, \Delta'_{(1245)})$ of distribution in the partial mapping f_3^2 if and only if when the coordinates of the tangent vector $\vec{\theta}$ of the line θ are satisfy the conditions (3.1).

Similar to the above, we consider the four dimensional distribution $\Delta_{(1345)} = (X, \vec{e}_1, \vec{e}_3, \vec{e}_4, \vec{e}_5)$. Then $f_3^2(\Delta_{(1345)}) = \Delta'_{(1345)} = (F_3^2, \vec{e}_1, \vec{e}_3, \vec{e}_4, \vec{e}_5)$. Let $\gamma \subset \Delta_{(1345)}$. The tangent vector of the line γ have a form:

$$\vec{\gamma} = \gamma^1 \vec{e}_1 + (\gamma^1 c_1^2 + \gamma^5 c_5^2 + \gamma^4 c_4^2) \vec{e}_2 + (\gamma^1 c_1^3 + \gamma^3 c_3^3 + \gamma^4 c_4^3 + \gamma^5 c_5^3) \vec{e}_3 + (\gamma^1 c_1^4 + \gamma^3 c_3^4 + \gamma^4 c_4^4 + \gamma^5 c_5^4) \vec{e}_4 + \gamma^5 \vec{e}_5,$$

where c_i^j - is j -th coordinate of vectors $\vec{e}_i$.

From the condition $\vec{\gamma}, \vec{\bar{\gamma}} \in \Delta_{(1345)}$ it follows that:

$$\gamma^1 c_1^3 + \gamma^3 + \gamma^4 c_4^3 + \gamma^5 c_5^3 = 0.$$

Taking into account formulas (1.10), we find:

$$\gamma^1 C_{321}^2 + \gamma^3 \left[1 + \frac{C_{323}^2}{(\Lambda_{32}^2)^2} \right] + \gamma^4 C_{324}^2 + \gamma^5 C_{325}^2 = 0. \quad (3.2)$$

If the coordinates of the tangent vector $\vec{\gamma}$ of the γ satisfy the condition (3.2), then the lines $\gamma, \bar{\gamma} = f_3^2(\gamma)$ are quasi-double of the pair $(\Delta_{(1345)}, \Delta'_{(1345)})$ in the partial mapping f_3^2 in Euclidean space E_5 .

Thus, the following theorem is proven:

Theorem 3.2. *The lines $\gamma \subset \Delta_{(1345)}$ and $\bar{\gamma} = f_3^2(\gamma)$ are quasi-double lines of the pair $(\Delta_{(1345)}, \Delta'_{(1345)})$ in the partial mapping f_3^2 if and only if the coordinates of the tangent vector $\vec{\gamma}$ of the line γ are satisfy the conditions (3.2).*

Now, consider the lines that belong to three-dimensional distributions. Let $\ell \subset \Delta_{(145)}$, and let the tangent vector of the line ℓ be given by: $\vec{\ell} = \ell^1 \vec{e}_1 + \ell^4 \vec{e}_4 + \ell^5 \vec{e}_5$. Then, under the mapping $f_3^2(\ell) = \bar{\ell}$ has a tangent vector $\vec{\bar{\ell}} = \ell^1 \vec{e}_1 + \ell^4 \vec{e}_4 + \ell^5 \vec{e}_5$.

Using formulas (1.10), we obtain:

$$\vec{\bar{\ell}} = \ell^1 \vec{e}_1 + (\ell^1 c_1^2 + \ell^4 c_4^2 + \ell^5 c_5^2) \vec{e}_2 + (\ell^1 c_1^3 + \ell^4 c_4^3 + \ell^5 c_5^3) \vec{e}_3 + (\ell^1 c_1^4 + \ell^4 c_4^4 + \ell^5 c_5^4) \vec{e}_4 + \ell^5 \vec{e}_5.$$

From the condition $\vec{\bar{\ell}}, \vec{\ell} \in \Delta_{(145)}$ we obtain:

$$\begin{cases} \ell^1 c_1^2 + \ell^4 c_4^2 + \ell^5 c_5^2 = 0; \\ \ell^1 c_1^3 + \ell^4 c_4^3 + \ell^5 c_5^3 = 0. \end{cases}$$

It follows that:

$$\ell^1 = \begin{vmatrix} c_4^2 & c_5^2 \\ c_4^3 & c_5^3 \end{vmatrix}; \quad \ell^4 = \begin{vmatrix} c_5^2 & c_1^2 \\ c_5^3 & c_1^3 \end{vmatrix}; \quad \ell^5 = \begin{vmatrix} c_1^2 & c_4^2 \\ c_1^3 & c_4^3 \end{vmatrix}.$$

Using formulas (1.10), we obtain:

$$\ell^1 = \begin{vmatrix} \Lambda_{34}^2 & \Lambda_{35}^2 \\ C_{324}^2 & C_{325}^2 \end{vmatrix}; \quad \ell^4 = \begin{vmatrix} \Lambda_{35}^2 & \Lambda_{31}^2 \\ C_{325}^2 & C_{321}^2 \end{vmatrix}; \quad \ell^5 = \begin{vmatrix} \Lambda_{31}^2 & \Lambda_{34}^2 \\ C_{321}^2 & C_{324}^2 \end{vmatrix}. \quad (3.3)$$

If the coordinates of the tangent vector $\vec{\ell}$ of $\ell \subset \Delta_{(145)}$ satisfy condition (3.3), then the lines $\ell, \bar{\ell} = f_3^2(\ell)$ are quasi-double lines of the pair $(\Delta_{(145)}, \Delta'_{(145)})$ in the partial mapping f_3^2 .

Now, let the line d belong to the three-dimensional distribution $\Delta_{(245)}$. Then its tangent vector $\vec{d}$ has the form: $\vec{d} = d^2 \vec{e}_2 + d^4 \vec{e}_4 + d^5 \vec{e}_5$. The tangent vector $\vec{\bar{d}}$ of the line $\bar{d} = f_3^2(d)$ has a form: $\vec{\bar{d}} = d^2 \vec{e}_2 + d^4 \vec{e}_4 + d^5 \vec{e}_5$.

Using formulas (1.10), we obtain:

$$\vec{\bar{d}} = (d^2 + d^4 c_4^2 + d^5 c_5^2) \vec{e}_2 + (d^2 c_2^3 + d^4 c_4^3 + d^5 c_5^3) \vec{e}_3 + (d^2 c_2^4 + d^4 + d^5 c_5^4) \vec{e}_4 + d^5 \vec{e}_5.$$

From the condition $\vec{\bar{d}}, \vec{d} \in \Delta_{(245)}$ it follows that:

$$d^2 c_2^3 + d^4 c_4^3 + d^5 c_5^3 = 0.$$

Taking into account formula (1.10) we obtain:

$$d^2 c_{322}^2 + d^4 c_{324}^2 + d^5 c_{325}^2 = 0. \quad (3.4)$$

If the coordinates of tangent vector $\vec{d}$ of $d \subset \Delta_{(245)}$ satisfy condition (3.4), then the lines $d, \bar{d} = f_3^2(d)$ are quasi-double lines of the pair $(\Delta_{(245)}, \Delta'_{(245)})$ in the partial mapping f_3^2 , where $\Delta'_{(245)} = f_3^2(\Delta_{(245)})$.

Let the line m belong to the three-dimensional distribution $\Delta_{(345)}$. Then its tangent vector $\vec{m}$ has the form: $\vec{m} = m^3 \vec{e}_3 + m^4 \vec{e}_4 + m^5 \vec{e}_5$. A tangent vector $\vec{\bar{m}}$ of the line $\bar{m} = f_3^2(m)$ has the form: $\vec{\bar{m}} = m^3 \vec{e}_3 + m^4 \vec{e}_4 + m^5 \vec{e}_5$. Using formulas (1.10) we obtain:

$$\vec{\bar{m}} = (m^4 c_4^2 + m^5 c_5^2) \vec{e}_2 + (m^3 c_3^3 + m^4 c_4^3 + m^5 c_5^3) \vec{e}_3 + (m^3 c_3^4 + m^4 + m^5 c_5^4) \vec{e}_4 + m^5 \vec{e}_5.$$

From the condition $\vec{m}, \vec{\bar{m}} \in \Delta_{(345)}$ it follows that:

$$m^4 c_4^2 + m^5 c_5^2 = 0.$$

Taking into account formula (1.10) we obtain:

$$m^4 \Lambda_{34}^2 + m^5 \Lambda_{35}^2 = 0. \quad (3.5)$$

If the coordinates of the tangent vector $\vec{m}$ of $m \subset \Delta_{(345)}$ satisfy condition (3.5), then the lines $m, \bar{m} = f_3^2(m)$ are quasi-double lines of the pair $(\Delta_{(345)}, \Delta'_{(345)})$ in the partial mapping f_3^2 , where $\Delta'_{(345)} = f_3^2(\Delta_{(345)})$.

Now, let the line s belong to the three-dimensional distribution $\Delta_{(123)} = (X, \vec{e}_1, \vec{e}_2, \vec{e}_3)$. Then its tangent vector $\vec{s}$ has the form: $\vec{s} = s^1 \vec{e}_1 + s^2 \vec{e}_2 + s^3 \vec{e}_3$. The tangent vector $\vec{\bar{s}}$ of the line $\bar{s} = f_3^2(s)$ has the form: $\vec{\bar{s}} = s^1 \vec{c}_1 + s^2 \vec{c}_2 + s^3 \vec{c}_3$. Using formulas (1.10) we obtain:

$$\vec{s} = s^1 \vec{e}_1 + (s^1 c_1^2 + s^2) \vec{e}_2 + (s^1 c_1^3 + s^2 c_2^3 + s^3 c_3^3) \vec{e}_3 + (s^1 c_1^4 + s^2 c_2^4 + s^3 c_3^4) \vec{e}_4.$$

From the condition $\vec{s}, \vec{\bar{s}} \in \Delta_{(123)}$ it follows that:

$$s^1 c_1^4 + s^2 c_2^4 + s^3 c_3^4 = 0.$$

Taking into account formula (1.10) we obtain:

$$s^1 \Lambda_{31}^4 + s^2 \Lambda_{32}^4 + s^3 \Lambda_{33}^4 = 0. \quad (3.6)$$

If the coordinates of the tangent vector $\vec{s}$ of the $s \subset \Delta_{(123)}$ satisfy condition (3.6), then the lines $s, \bar{s} = f_3^2(s)$ are quasi-double lines of the pair $(\Delta_{(123)}, \Delta'_{(123)})$ in the partial mapping f_3^2 , where $\Delta'_{(123)} = f_3^2(\Delta_{(123)})$.

From the above it follows:

Theorem 3.3. *a) The lines $\ell \subset \Delta_{(145)}$ and $\bar{\ell} = f_3^2(\ell)$ are quasi-double lines of the pair $(\Delta_{(145)}, \Delta'_{(145)})$ in the partial mapping f_3^2 if and only if the coordinates of the tangent vector $\vec{\ell}$ of the line ℓ are satisfy the conditions (3.3);*

b) The lines $d \subset \Delta_{(245)}$ and $\bar{d} = f_3^2(d)$ are quasi-double lines of the pair $(\Delta_{(245)}, \Delta'_{(245)})$ in the partial mapping f_3^2 if and only if the coordinates of the tangent vector $\vec{d}$ of the line d are satisfy the conditions (3.4);

c) The lines $m \subset \Delta_{(345)}$ and $\bar{m} = f_3^2(m)$ are quasi-double lines of the pair $(\Delta_{(345)}, \Delta'_{(345)})$ in the partial mapping f_3^2 if and only if the coordinates of the tangent vector $\vec{m}$ of the line m satisfy conditions (3.5);

d) The lines $s \subset \Delta_{(123)}$ and $\bar{s} = f_3^2(s)$ are quasi-double lines of the pair $(\Delta_{(123)}, \Delta'_{(123)})$ in the partial mapping f_3^2 if and only if the coordinates of the tangent vector $\vec{s}$ of the line s are satisfy conditions (3.6).

4. Conclusion

Pairs of three-dimensional distributions in the space E_5 are considered.

Necessary and sufficient conditions are established for the lines belonging to a three-dimensional distribution to be quasi-double lines of pairs of distributions under the pairwise mapping of the Euclidean space E_5 .

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