

Compactness of Uniform Spaces

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Abstract

This paper introduces u -uniformly μ -compact spaces and the u -uniform index of compactness of uniform spaces and investigates their connections with other uniform properties of compactness type.

Keywords: u -uniformly compactness, u -uniformly μ -compactness, u -uniformly index compactness, u -uniformly compactness, u -uniformly finally compactness, u -uniformly countable compactness.

2010 MSC: 54E15, 54D20.

1. Introduction

Throughout this paper all uniform spaces are assumed to be Hausdorff, topological spaces are Tychonoff and mappings are uniformly continuous.

It is well known that compactness, μ -compactness and index of compactness of topological spaces play an important role in set-theoretic topology. Finding and studying uniform analogues of these classes of compactness types is an important and interesting problem in Uniform Topology.

A topological space X is called countable compact if every countable open cover has a finite open refinement [1]; a topological space X is called μ -compact if every open cover of cardinality $\leq \mu$ has a finite open refinement; if $\mu \leq \aleph_0$, then the topological space X is called countable compact [3]; a topological space X is called finally compact if every open cover has a countable open refinement [1]; a topological space X has the index of compactness $ic(X) \leq \mu$, if every open cover has a $\aleph$ open refinement of cardinality $\leq \mu$; if $ic(X) \leq \aleph_0$, then the topological space X is called finally compact [3]; a uniform space (X, U) is called precompact if it has a base consisting of finite covers [2], [3], [11]; a uniform space (X, U) is called strongly uniformly paracompact if every finitely additive open cover has a star finite uniform refinement [3], [8], [12]; a uniform space (X, U) is called complete if every Cauchy filter in it converges, where a filter F is called a Cauchy filter if $\alpha \cap F \neq \emptyset$ for any $\alpha \in U$ [6], [7]; a uniformly continuous mapping $f : (X, U) \rightarrow (Y, V)$ of a uniform space (X, U) onto a uniform space (Y, V) is called precompact, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and a finite uniform cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \geq \alpha$ [1]; a uniformly continuous mapping $f : (X, U) \rightarrow (Y, V)$ of a uniform space (X, U) onto a uniform space (Y, V) is called uniformly perfect, if it is precompact and perfect [1]; a continuous mapping $f : X \rightarrow Y$ of a topological space X to a topological space Y is called an ω -mapping if for each point $y \in Y$ there exist neighborhood O_y and $W \in \omega$, such that $f^{-1}O_y \subset W$ [4].

For a cover α of a set X and $x \in X$, $M \subset X$, we have: $St(\alpha, x) = \{A \in \alpha : x \in A\}$, $\alpha(x) = \cup St(\alpha, x)$, $St(\alpha, M) = \{A \in \alpha : A \cap M \neq \emptyset\}$, $\alpha(M) = \cup St(\alpha, M)$. For covers α and β of the set X , the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the

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cover β , i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$, and for covers α and β of a set X , we have: $\alpha \wedge \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. For a cover α of a set X let $\alpha^\angle = \{\cup \alpha_0 : \alpha_0 \subset \alpha - \text{is finite}\}$. The cover α is finitely additive if $\alpha^\angle = \alpha$. For a uniformity U by τ_U we denote the topology generated by the uniformity, and the symbol U_X means the universal uniformity.

2. Compactness of Uniform Spaces

In the paper [13], u -uniformly compact, u -uniformly countably compact and u -uniformly finally compact spaces were introduced and studied. This paper introduces and studies uniform spaces (X, U) with $u - ic(X) \leq \mu$ and u -uniformly μ -compact spaces.

The following lemma will serve as the basis for our definitions.

A topological space (X, τ) is compact (μ -compact, has $ic(X) \leq \mu$) if and only if every finitely additive (of cardinality $\leq \mu$) open cover (of cardinality $\leq \mu$) has an open finite (finite, of cardinality $\leq \mu$) refinement.

Proof. Let α be an arbitrary (of cardinality $\leq \mu$) open cover of (X, τ) . Then for the finitely additive (of cardinality $\leq \mu$) open cover $\alpha^\angle$ there exists a finite (finite, of cardinality $\leq \mu$) open cover β such that $\beta \succ \alpha^\angle$. For each $G \in \gamma$ choose $A_G \in \alpha^\angle$ such that $G \subset A_G$, $A_G = \bigcup_{i=1}^n A_i$, $A_i \in \alpha$, $i = 1, 2, \dots, n$. Let $\alpha_0 = \bigcup \{\alpha_G : G \in \gamma\}$, $\alpha_G = \{G \cap A_i : i = 1, 2, \dots, n\}$. Then α_0 is finite (finite, of cardinality $\leq \mu$) open cover of the topological space (X, τ) such that $\alpha_0 \succ \alpha$. Thus, the topological space (X, τ) is compact (μ -compact, has $ic(X) \leq \mu$).

The converse is obvious. $\square$

Definition 2.1. A uniform space (X, U) is called u -uniformly compact (u -uniformly μ -compact, has $u - ic(X) \leq \mu$) if every finitely additive (of cardinality $\leq \mu$) open cover has a finite (finite, of cardinality $\leq \mu$) uniform refinement.

Proposition 2.1. If a uniform space (X, U) is u -uniformly compact (u -uniformly μ -compact, has $u - ic(X) \leq \mu$), then topological space (X, τ_U) is compact (μ -compact, has $ic(X) \leq \mu$). Conversely, if a topological space (X, τ) is compact (μ -compact, has $ic(X) \leq \mu$), then the space (X, U_X) is compact (u -uniformly μ -compact, has $u - ic(X) \leq \mu$), where U_X is the universal uniformity on topological space (X, τ) such that $\tau_{U_X} = \tau$.

Proof. Let α be an arbitrary (of cardinality $\leq \mu$) open cover of the space (X, τ_U) . Then for the finitely additive (of cardinality $\leq \mu$) open cover $\alpha^\angle$ of (X, U) there exist finite (finite, of cardinality $\leq \mu$) uniform cover $\beta \in U$ such that $\beta \succ \alpha^\angle$. It is known that the interior $\langle \beta \rangle$ of the uniform cover β is a uniform cover. Denote $\gamma = \langle \beta \rangle$. Then γ is an finite (finite, of of cardinality $\leq \mu$) open uniform cover of the uniform space (X, U) . For each $G \in \gamma$ we choose $A_G \in \alpha^\angle$ such that $G \subset A_G$, $A_G = \bigcup_{i=1}^n A_i$, $A_i \in \alpha$, $i = 1, 2, \dots, n$. We put $\alpha_0 = \bigcup \{\alpha_G : G \in \gamma\}$, $\alpha_G = \{G \cap A_i : i = 1, 2, \dots, n\}$. Then α_0 is finite (finite, of cardinality $\leq \mu$) open cover of the space (X, τ_U) and $\alpha_0 \succ \alpha$. Thus, (X, τ_U) is compact (μ -compact, has $ic(X) \leq \mu$).

Conversely, let (X, τ) be compact (μ -compact, has $ic(X) \leq \mu$). Then the set of all open covers form the base of the universal uniformity U_X of the topological space (X, τ) . It is evident that the uniform space (X, U_X) is u -uniformly compact (u -uniformly μ -compact, has $u - ic(X) \leq \mu$).

Obviously, every u -uniformly compact space is strongly uniformly paracompact. $\square$

Corollary 2.1. If a uniform space (X, U) is u -uniformly countably compact (u -uniformly finally compact), then the topological space (X, τ_U) is countably compact (finally compact). Conversely, if a topological space (X, τ) is countably compact (finally compact), then the space (X, U_X) is u -uniformly countably compact (u -uniformly finally compact), where U_X is the universal uniformity on the topological space (X, τ) such that $\tau_{U_X} = \tau$.

The following theorem gives a characterization of the u -uniform index of compactness in the spirit of Tamano.

Theorem 2.1. Let (X, U) be a uniform space and bX be a compact Hausdorff extensions of the space (X, τ_U) . The uniform space (X, U) has $u - ic(X) \leq \mu$ if and only if for a compact $K \subset bX \setminus X$ there exist a uniform cover $\alpha \in U$ of cardinality $\leq \mu$, such that $[A]_{bX} \cap K = \emptyset$ for any $A \in \alpha$.

Proof. Necessity. Let (X, U) be $u - ic(X) \leq \mu$ and $K \subset bX \setminus X$ be an arbitrary compact set. For each point $x \in X$ there is an open neighborhood O_x of the compact Hausdorff extensions bX of the space (X, τ_U) , such that $[O_x]_{bX} \cap K = \emptyset$. Put $\gamma = \{O_x \cap X : x \in X\}$. Then γ is an open cover of the space (X, U) . Let $\beta \in U$ be a uniform cover of the space (X, U) of cardinality $\leq \mu$ such that $\beta \succ \gamma^\angle$, then $[B]_{bX} \subset [\cup_{i=1}^n (O_{x_i} \cup X)]_{bX} \subset \cup_{i=1}^n [O_{x_i}]_{bX}$. Since $[O_{x_i}]_{bX} \cap K = \emptyset$, $i = 1, 2, \dots, n$, then $[B]_{bX} \cap K = \emptyset$ for each $B \in \beta$. $[B]_{bX} \cap K = \emptyset$ for each $B \in \beta$.

Sufficiency. Let α be an arbitrary finitely additive open cover of the space (X, U) . Then there exists an open family β of the compact Hausdorff extensions bX , such that $\beta \wedge \{X\} = \alpha$. Put $K = bX \cup \beta$. It is easy to see that K is compact. Accordingly, by the condition of the theorem, there exists a cover $\gamma \in U$ of cardinality $\leq \mu$ such that $[G]_{bX} \cap K = \emptyset$, $G \in \gamma$. Since $[G]_{bX}$ is compact in bX , there exist $B_1, B_2, \dots, B_n \in \beta$, such that $[G]_{bX} \subset \cup_{i=1}^n B_i$. Then $G \subset \cup_{i=1}^n A_i$, $\cup_{i=1}^n A_i \in \alpha$. Consequently, $u - ic(X) \leq \mu$. $\square$

Corollary 2.2. Let (X, U) be a uniform space and bX be a compact Hausdorff extensions of the space (X, τ_U) . A uniform space (X, U) is u -uniformly compact (u -uniformly finally compact) if and only if for a compact set $K \subset bX \setminus X$ there exist a finite (of cardinality $\leq \mu$) uniform cover $\alpha \in U$, such that $[A]_{bX} \cap K = \emptyset$ for each $A \in \alpha$.

Proposition 2.2. Any u -uniformly compact uniform space (X, U) satisfies $u - ic(X) \leq \mu$ (is u -uniformly μ -compact).

Proof. Let α be an arbitrary (of cardinality $\leq \mu$) open cover of (X, U) . Since the space (X, U) is u -uniformly compact, there exists a finite uniform cover $\beta \in U$ such that $\beta \succ \alpha$. Thus, (X, U) has $u - ic(X) \leq \mu$ (is u -uniformly μ -compact). $\square$

Corollary 2.3. Any u -uniformly compact uniform space (X, U) is u -uniformly finally compact (u -uniformly countably compact).

Proposition 2.3. If $u - ic(X) \leq \mu$, then (X, U) is complete.

Proof. Let (X, U) be a u -uniformly compact space and F be an arbitrary Cauchy filter. Suppose that F does not converge to any point of the space (X, U) . Then each point $x \in X$ has an open neighborhood O_x such that $O_x \cap B = \emptyset$ for a $B \in F$. Denote $\alpha = \{O_x : x \in X\}$. Since $u - ic(X) \leq \mu$, then $\alpha^\angle \in U$. Therefore $\alpha^\angle \cap F \neq \emptyset$, i.e. there is $A^\angle \in \alpha^\angle$ such that $A^\angle \in F$, where $A^\angle = \cup_{i=1}^n O_{x_i}$, $O_{x_i} \in \alpha$. Hence $(\cap_{i=1}^n B_{x_i}) \cap (\cup_{i=1}^n O_{x_i}) \neq \emptyset$. Then, there exists $i_0 \leq n$, such that $B_{x_{i_0}} \cap O_{x_{i_0}} \neq \emptyset$. A contradiction. Thus, (X, U) is complete. $\square$

Corollary 2.4. Any u -uniformly finally compact uniform space (X, U) is complete.

Corollary 2.5. Any u -uniformly compact uniform space (X, U) is complete.

In the class of precompact uniform spaces compactness, u -uniform compactness and u -uniform final compactness are equivalent.

Proposition 2.4. The real line R with the natural uniformity has $u - ic(X) \leq \mu$.

Proof. Let α be an arbitrary finitely additive cover of R and $\beta = \{(n-1, n+1) : n = 0, \pm 1, \pm 2, \dots\}$. Then, $\beta \in U$ is a uniform cover of cardinality $\leq \mu$ of R . Since $[n-1, n+1]$ is compact, there exists a finite family $\{A_1, A_2, \dots, A_n\}$, $A_i \in \alpha$, $i = 1, 2, \dots, n$, such that $(n-1, n+1) \subset [n-1, n+1] \subset \bigcup_{i=1}^n A_i$, i.e. $\beta \succ \alpha^\angle = \alpha$. Hence, R has $u - ic(X) \leq \mu$, but it is not u -uniformly compact. $\square$

Corollary 2.6. The real line R with the natural uniformity is u -uniformly finally compact.

Let (X, r) be a μ -compact, but not compact space. Then the uniform space (X, U_X) , according to Proposition 2.1, is u -uniformly μ -compact but not u -uniformly compact.

u -uniformly compactness (u -uniformly μ -compactness, property $u - ic(X) \leq \mu$) is inherited by taking closed subspaces.

Theorem 2.2. Any closed subspace M of u -uniformly μ -compact (space with $u - ic(X) \leq \mu$) space (X, U) is u -uniformly μ -compact (has $u - ic(M) \leq \mu$),

Proof. Let γ_M be an arbitrary finitely additive (of cardinality $\leq \mu$) open cover of the subspace (M, U_M) . Then there exists a finitely additive (of cardinality $\leq \mu$) open family γ of the space (X, U) such that $\gamma \wedge \{M\} = \gamma_M$. Denote $\hat{\gamma} = \{\gamma, X \setminus M\}$. Obviously the cover $\hat{\gamma}$ is a finitely additive (if cardinality $\leq \mu$) open cover of the space (X, U) . Then, there exists a finite (of cardinality $\leq \mu$) uniform cover $\beta \in U$, such that $\beta \succ \hat{\gamma}$. Put $\beta_M = \beta \wedge \{M\}$. It is easy to see that β_M is finite (of cardinality $\leq \mu$) uniform cover of the subspace (M, U_M) and $\beta_M \succ \gamma$. Hence, (M, U_M) is u -uniformly μ -compact, ($u - ic(M) \leq \mu$). $\square$

Corollary 2.7. Any closed subspace M of u -uniformly compact (u -uniformly finally compact, u -uniformly countably compact) space (X, U) is u -uniformly compact (u -uniformly finally compact, u -uniformly countably compact) space.

Theorem 2.3. Let $f : (X, U) \rightarrow (Y, V)$ be an ω -mapping between uniform spaces (X, U) and (Y, V) . If (Y, V) is u -uniformly μ -compact (has $u - ic(Y) \leq \mu$), then the uniform space (X, U) is also u -uniformly μ -compact (has $u - ic(X) \leq \mu$).

Proof. Let ω be an arbitrary finitely additive open cover of the space (X, U) of cardinality $\leq \mu$ and $f : (X, U) \rightarrow (Y, V)$ be an ω -mapping of the uniform space (X, U) onto u -uniformly μ -compact (space with $u - ic(Y) \leq \mu$) space (Y, V) . Then, for each point $y \in Y$ there exist a neighborhood O_y and $W \in \omega$, such that $f^{-1}O_y \subset W$. Denote $\beta = \{O_y : y \in Y\}$, $\beta^\angle = \{\bigcup \beta_0 : \beta_0 \subset \beta \text{ is finite}\}$. Since the space (Y, V) is u -uniformly μ -compact (has $u - ic(Y) \leq \mu$), there exists a finite (of cardinality $\leq \mu$) uniform cover $\gamma \in V$, such that $\gamma \succ \beta^\angle$. Then finite (of cardinality $\leq \mu$) uniform cover $f^{-1}\gamma$ refines ω . Thus, (X, U) is u -uniformly μ -compact ($u - ic(X) \leq \mu$). $\square$

Corollary 2.8. Let $f : (X, U) \rightarrow (Y, V)$ be an ω -mapping between uniform spaces (X, U) and (Y, V) . If (Y, V) is a u -uniformly compact (u -uniformly finally compact, u -uniformly countably compact) space, then the uniform space (X, U) is also u -uniformly compact (u -uniformly finally compact, u -uniformly countably compact).

Theorem 2.4. Let $f : (X, U) \rightarrow (Y, V)$ be a perfect mapping between uniform spaces (X, U) and (Y, V) . If (Y, V) is u -uniformly μ -compact ($u - ic(Y) \leq \mu$), then the uniform space (X, U) is also u -uniformly μ -compact ($u - ic(X) \leq \mu$).

Proof. Let α be an arbitrary finitely additive open cover of cardinality $\leq \mu$ of the uniform space (X, U) . It is clear that the cover $\{f^{-1}y : y \in Y\}$ is a refinement of the cover α . Then, $\beta = f^*\alpha = \{f^*A : A \in \alpha\}$, where $f^*A = Y \setminus f(X \setminus A)$ is an open cover of the space (Y, V) . Since the space (Y, V) is u -uniformly μ -compact (satisfies $u - ic(Y) \leq \mu$), there exists a finite (of cardinality $\leq \mu$) cover $\gamma \in V$ such that $\gamma \succ \beta^\angle$. It is easy to see that $f^{-1}\gamma \succ \alpha$ and $f^{-1}\gamma$ is a finite (of cardinality $\leq \mu$) cover of the space (X, U) . Consequently, the space (X, U) is u -uniformly μ -compact (satisfies $u - ic(X) \leq \mu$). $\square$

Corollary 2.9. Let $f : (X, U) \rightarrow (Y, V)$ be a perfect mapping between uniform spaces (X, U) and (Y, V) . If (Y, V) is u -uniformly compact (u -uniformly finally compact, u -uniformly countably compact) space, then the uniform space (X, U) is also u -uniformly compact (u -uniformly finally compact, u -uniformly countably compact).

Corollary 2.10. Let $f : (X, U) \rightarrow (Y, V)$ be a uniformly perfect mapping between uniform spaces (X, U) and (Y, V) . If (Y, V) is u -uniformly μ -compact (satisfies $u - ic(Y) \leq \mu$), then the uniform space (X, U) is also u -uniformly μ -compact (satisfies $u - ic(X) \leq \mu$).

Proposition 2.5. Product $(X, U) \times (Y, V)$ of a u -uniformly μ -compact space (space with $u - ic(X) \leq \mu$) (X, U) by a compact space (Y, V) is u -uniformly μ -compact (satisfies $u - ic(X \times Y) \leq \mu$).

Proof. Let (X, U) be a u -uniformly μ -compact space (space with $u - ic(X) \leq \mu$) and (Y, V) be a compact space. It is known that projection $\pi_X : (X, U) \times (Y, V) \rightarrow (X, U)$ is uniformly perfect mapping of the product $(X, U) \times (Y, V)$ onto the u -uniformly μ -compact space (space with $u - ic(Y) \leq \mu$) (Y, V) . Then, the projection π_X is an ω -mapping for any finitely additive (of cardinality $\leq \mu$) open cover ω . Then, according to Theorem 2.3., the product $(X, U) \times (Y, V)$ is u -uniformly μ -compact (satisfies $u - ic(X \times Y) \leq \mu$). $\square$

Corollary 2.11. The product $(X, U) \times (Y, V)$ of a u -uniformly compact (u -uniformly countably compact, u -uniformly finally compact) space (X, U) by a compact space (Y, V) is u -uniformly compact (u -uniformly countably compact, u -uniformly finally compact).

Theorem 2.5. A uniform space (X, U) is u -uniformly compact if and only if (X, U) is u -uniformly μ -compact and $u - ic(X) \leq \mu$.

Proof. Necessity follows from Proposition 2.2.

Sufficiency. Let α be an arbitrary finitely additive open cover of the space (X, U) . Since the space (X, U) is $u - ic(X) \leq \mu$, there exists a cover $\beta \in U$ of cardinality $\leq \mu$ such that $\beta \succ \alpha$. Since the space (X, U) is u -uniformly μ -compact there exists a finite uniform cover $\gamma \in U$, such that $\gamma \succ \beta^\angle$. Then, $\gamma \succ \beta^\angle \succ \alpha^\angle = \alpha$, which means that $\gamma \succ \alpha$. Thus, (X, U) is u -uniformly compact. $\square$

Corollary 2.12. A uniform space (X, U) is u -uniformly compact if and only if (X, U) is u -uniformly finally compact and u -uniformly countably compact.

Corollary 2.13. A Tychonoff space (X, τ) is compact if and only if (X, τ) is finally compact and countably compact.

3. Conflict of interests

The authors declare that have no conflicts of interest.

4. Acknowledgments

The corrections and excellent suggestions of the anonymous reviewer are gratefully acknowledged.

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