

Construction of the Solution to a Second-Order Partial Differential Equation of Parabolic Type with Multiple Variables by the Method of Additional Argument

A.Zh. Ashirbaeva^{*1}, M. Bilal kzy², G.A. Karimova³, G.K. Sadykova⁴

^{1,2} *Osh Technological University named after M. Adyshev, Osh, Kyrgyzstan*

³ *Jusup Balasagyn Kyrgyz National University, Bishkek, Kyrgyzstan*

⁴ *Osh State University, Osh, Kyrgyzstan*

Abstract

The article examines a parabolic-type partial differential equation. It is well known that the phenomenon of heat transfer in physics is modeled by parabolic-type equations. Therefore, the problems of constructing solutions to such equations are of both theoretical and practical interest. In our work there is considered the case of multiple variables and equation of second order. The purpose of the work is to apply the method of additional argument to parabolic-type equations. The Cauchy-Dirichlet problem is investigated, and its unique solution is established. The obtained result is illustrated by a concrete example. Similar results can be extended to other types of partial differential equations.

Keywords: parabolic type, partial derivative, multidimensional, additional argument method, solution, construction, uniqueness.

2010 MSC: 35C15.

1. Introduction

The method of additional argument (MAA) has not yet gained widespread use for parabolic equations. This method allows reducing partial differential equations to integral ones. However, it is not applicable to all partial differential equations. In some cases, the equation can be transformed into a form suitable for applying this method. The method has been successfully employed for various hyperbolic equations [1,3-10]. The application of this method to parabolic equations with two variables was considered in the paper [2]. In this paper we generalize the equation to the multidimensional case.

The MAA can also be regarded as a development of the method of characteristics. When the partial differential equation is nonlinear, the application of this method yields good results. In paper [5] a solution to a nonlinear partial differential equation was constructed using this method.

The application of the MAA to systems of partial differential equations also yields good results. For instance, in paper [1], a solution to a system of nonlinear equations was constructed. The application of the MAA to higher-order partial differential equations, as well as results obtained for the case of multiple variables, can be found in [4,10].

^{*}Corresponding author: ajjarkyn.osh@mail.ru

2. Formulation of the Problem

In the present work, the problem of solving a second-order parabolic-type partial differential equation (PDE) with $(n+1)$ independent variables using the method of additional argument (MAA) is considered.

The PDE is considered in operator form:

$$D^2[a_1(t, x_1), a_2(t, x_2), \dots, a_n(t, x_n)]u(t, x_1, x_2, \dots, x_n) = f(t, x_1, x_2, \dots, x_n), \quad (2.1)$$

$$(t, x_1, x_2, \dots, x_n) \in Q_2(T) = \{0 \leq t \leq T, x = (x_1, x_2, \dots, x_n) \in R^n\},$$

where operator:

$$D[\omega_1, \omega_2, \dots, \omega_n] = \partial/\partial t + \omega_1 \partial/\partial x_1 + \dots + \omega_n \partial/\partial x_n,$$

the given functions $a_1(t, x_1), a_2(t, x_2), \dots, a_n(t, x_n), f(t, x_1, x_2, \dots, x_n)$ are continuous and bounded, and possess continuous and bounded partial derivatives by all their arguments.

In particular, if $n = 2$, then equation (2.1) takes the form:

$$D^2[a(t, x), b(t, y)]u(t, x, y) = u_{tt} + 2au_{tx} + 2bu_{ty} + 2abu_{xy} + a^2u_{xx} + b^2u_{yy} +$$

$$+(a_t + aa_x)u_x + (b_t + bb_y)u_y = f(t, x, y). \quad (2.2)$$

The Cauchy-Dirichlet problem is considered:

$$u(x_1, p_1(x_1, 0, x_1), p_2(x_1, 0, x_2), \dots, p_n(x_1, 0, x_n)) = \psi(x_1, x_2, \dots, x_n), \quad (2.3)$$

where $p_1(s, t, x_1), p_2(s, t, x_2), \dots, p_n(s, t, x_n)$ are the corresponding solutions of the following integral equations:

$$p_k(s, t, x_k) = x_k - \int_s^t a_k(\nu, p_k(\nu, t, x_k)) d\nu, \quad k = 1, \dots, n. \quad (2.4)$$

Since the given functions $a_1(t, x_1), a_2(t, x_2), \dots, a_n(t, x_n)$ are sufficiently smooth, that is, together with their derivatives they belong to the space of continuous and bounded functions, it follows from the theory of integral equations that the solution of the integral equation (2.4) exists and is unique [5]. By applying differentiation to the integral equation (2.4), we obtain the following equalities:

$$\partial p_k(s, t, x)/\partial t + a_k(t, x_k) \partial p_k(s, t, x_k)/\partial x_k = 0, \quad p(s, s, x_k) = x_k, \quad k = 1, \dots, n.$$

3. Materials and Methods of Study

Theorem 3.1. *Let the functions given in (2.1) and (2.3) be continuous and possess continuous and bounded partial derivatives up to and including the second order. Then the Cauchy-Dirichlet problem (2.1), (2.3) has a unique solution.*

Proof. The solutions of equation (2.1) are obtained by reducing the order of the differential equation using the method of additional argument (MAA):

$$D[a_1(t, x_1), a_2(t, x_2), \dots, a_n(t, x_n)]u(t, x_1x_2, \dots, x_n) = \varphi_1(p_1(0, t, x_1), p_2(0, t, x_2), \dots, p_n(0, t, x_n)) + \int_0^t f(\nu, p_1(\nu, t, x_1), p_2(\nu, t, x_2), \dots, p_n(\nu, t, x_n)) d\nu, \quad (3.1)$$

$$(t, x_1, x_2, \dots, x_n) \in Q_2(T).$$

We apply the MAA once again, now to equation (3.1):

$$\begin{aligned} u(t, x_1x_2, \dots, x_n) &= \varphi_1(p_1(0, t, x_1), p_2(0, t, x_2), \dots, p_n(0, t, x_n))t + \\ &+ \varphi_2(p_1(0, t, x_1), p_2(0, t, x_2), \dots, p_n(0, t, x_n)) + \\ &+ \int_0^t (t - \nu) f(\nu, p_1(\nu, t, x_1), p_2(\nu, t, x_2), \dots, p_n(\nu, t, x_n)) d\nu, \end{aligned} \quad (3.2)$$

$$(t, x_1, x_2, \dots, x_n) \in Q_2(T).$$

The function (3.2) is the general solution of equation (2.1). In the subsequent calculations, we use the following equation, proved by A.Zh. Ashirbaeva [1]:

$$p_i(\tau, t, p_i(t, \theta, x)) = p_i(\tau, \theta, x), \quad (\tau, t, \theta, x) \in Q_3(T),$$

$$Q_3(T) = \{(\tau, t, \theta, x) | 0 \leq \tau \leq t \leq \theta \leq T, x \in R\}, \quad i = 1, \dots, n.$$

The $\varphi_1(x_1, x_2, \dots, x_n)$, $\varphi_2(x_1, x_2, \dots, x_n)$ functions in (3.2) are arbitrary and we determine them using condition (2.3).

$$\begin{aligned} &u(x_1, p_1(x_1, 0, x_1), p_2(x_1, 0, x_2), \dots, p_n(x_1, 0, x_n)) = \\ &= \varphi_1(p_1(0, x_1, p_1(x_1, 0, x_1)), p_2(0, x_1, p_2(x_1, 0, x_2)), \dots, p_n(0, x_1, p_n(x_1, 0, x_n)))x_1 + \\ &+ \varphi_2(p_1(0, x_1, p_1(x_1, 0, x_1)), p_2(0, x_1, p_2(x_1, 0, x_2)), \dots, p_n(0, x_1, p_n(x_1, 0, x_n))) + \\ &+ \int_0^t (t - \nu) f(\nu, p_1(\nu, x_1, p_1(x_1, 0, x_1)), p_2(\nu, x_1, p_2(x_1, 0, x_2)), \dots, p_n(\nu, x_1, p_n(x_1, 0, x_n))) d\nu \end{aligned}$$

and we get:

$$\varphi_1(x_1, x_2, \dots, x_n) + \varphi_2(x_1, x_2, \dots, x_n)x_1 + \int_0^{x_1} (x_1 - \nu) f(\nu, p_1(\nu, 0, x_1), p_2(\nu, 0, x_2), \dots, p_n(\nu, 0, x_n)) d\nu = \psi(x_1, x_2, \dots, x_n).$$

Let the arbitrary functions be equal to: $\varphi_1(x_1, x_2, \dots, x_n) = \varphi_2(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n)$, and let $x_1 \neq -1$, then we obtain:

$$\begin{aligned} \varphi(x_1, x_2, \dots, x_n) = & \frac{1}{1+x_1} [\psi_1(x_1, x_2, \dots, x_n) - \\ & - \int_0^{x_1} (x_1 - \nu) f(\nu, p_1(\nu, 0, x_1), p_2(\nu, 0, x_2), \dots, p_n(\nu, 0, x_n)) d\nu]. \end{aligned} \quad (3.3)$$

By substituting (3.3) into (3.2), we obtain the solution to the Cauchy-Dirichlet problem (2.1), (2.3). The theorem is proved. $\square$

Let us consider the case of three independent variables and with coefficients equal to unity:

$$\begin{aligned} u_{tt}(t, x, y) + 2u_{tx}(t, x, y) + 2u_{ty}(t, x, y) + 2u_{xy}(t, x, y) + u_{xx}(t, x, y) + \\ + u_{yy}(t, x, y) = t \cdot x \cdot y \end{aligned} \quad (3.4)$$

For equation (3.4), we write the solutions of equation (2.4):

$$p_1(s, t, x) = x - t + s,$$

$$p_2(s, t, y) = y - t + s.$$

Equation (3.4) is considered with the condition:

$$u(x, 2x, y + x) = x + y, \quad (3.5)$$

For equation (3.4), we apply the MAA twice consecutively and obtain the general solution in the form:

$$\begin{aligned} u(t, x, y) = & \varphi_1(x - t, y - t)t + \varphi_2(x - t, y - t) + \\ & + \int_0^t (t - \nu) \nu (x - t + \nu)(y - t + \nu) d\nu, \end{aligned}$$

After integration, we obtain:

$$\begin{aligned} u(t, x, y) = & \varphi_1(x - t, y - t)t + \varphi_2(x - t, y - t) + \\ & + \frac{t^3}{6}(x - t)(y - t) + \frac{t^4}{12}(x - t) + \frac{t^4}{12}(y - t) + \frac{t^5}{20}. \end{aligned}$$

We determine the arbitrary functions from the boundary condition (3.4)

$$u(x, 2x, y + x) = \varphi_1(x, y)x + \varphi_2(x, y) + \frac{1}{4}x^4y + \frac{2}{15}x^5 = x + y,$$

Let: $\varphi_1(x, y) = \varphi_2(x, y) = \varphi(x, y)$.

Then $\varphi(x, y) = [x + y - \frac{1}{4}x^4y - \frac{2}{15}x^5]/(1 + x)$.

Consequently, the solution of the Cauchy-Dirichlet problem takes the form:

$$u(t, x, y) = [x - t + y - t - \frac{1}{4}(x - t)^4(y - t) - \frac{2}{15}(x - t)^5] \frac{1+t}{1+x-t} +$$

$$+ \frac{t^3}{6}(x-t)(y-t) + \frac{t^4}{12}(x-t) + \frac{t^4}{12}(y-t) + \frac{t^5}{20}.$$

The theorem on the existence and uniqueness of the solution to the Cauchy–Dirichlet problem (2.1), (2.3) has been proved by the method of additional argument. A solution has been constructed for the specific problem (3.4), (3.5).

4. Conclusion

The method of additional argument (MAA) has a number of advantages when constructing solutions to partial differential equations. The method of characteristics allows one to obtain a solution by reducing the equation to canonical form, whereas constructing a solution in the nonlinear case presents considerable difficulties. The results obtained in the paper show that the functions presented in equation (2.1) can be generalized to the case of n variables.

References

- [1] A.Zh. Ashirbaeva, *On the theory of higher-order integro-differential equations in partial derivatives of hyperbolic type*, Studies on Integro-Differential Equations. -Bishkek: Ilim, **32**(2003), 80–84.
- [2] A.Zh. Ashirbaeva, M. Bilal kyzy, *Construction of the solution of parabolic type by the method of additional argument*, Bulletin of Osh Technological University, **2**(2024), 90–93.
- [3] A.Zh. Ashirbaeva, E.A. Mamaziaeva, *Solution of a nonlinear operator-differential equation in partial derivatives of the second order of hyperbolic type*, Science and new technologies, **2**(2015), 8–11.
- [4] A.Zh. Ashirbaeva, G.K. Sadykova, *Solution of a system of nonlinear partial differential equations with $n + 1$ independent variables*, Eurasian Scientific Association, **8**(2021), 6–8.
- [5] A. Ashirbaeva, G. Sadykova, E. Mamaziaeva, Zh. Mambetov, *Solving of System of Partial Operator Equations of the First Order by the Method of Additional Argument*, Published by AIP Publishing. International Conference on Analysis and Applied Mathematics, ICAAM, AIP Conf. Proc., **3085**(2022), 020019-1–020019-8.
- [6] A.Zh. Ashirbaeva, Ch.B. Zholdosheva, *Solution of a nonlinear integro-differential equation in partial derivatives of the second order of hyperbolic type*, Bulletin of Osh State University. Series of natural and medical sciences, **2**(2012), 144–149.
- [7] A.Zh. Ashirbaeva, Ch. Zholdoshova, E. Mamaziaeva, *The researching of solutions of operator-differential equations in partial derivatives of the fourth order of hyperbolic type by the method of additional argument*, AIP Conf. Proc., **3244**(2024), 020044.
- [8] M.I. Imanaliev, P.S. Pankov, T.M. Imanaliev, *The method of additional argument in the theory of nonlinear wave partial differential equations*, Doklady of the Russian Academy of Sciences, **5**(1995), 596–598.
- [9] M.L. Krasnov, *Integral Equations*, -Moscow: Nauka, (1975).

- [10] G. K. Sadykova, *Construction of the solution to a system of nonlinear integro-differential equations in partial derivatives*, Science, New Technologies and Innovations of Kyrgyzstan, **7**(2021), 10–13.